NATURE OF METACOGNITION IN A DYNAMIC GEOMETRY ENVIRONMENT

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Abstract This case study examined the metacognitive processes of a preservice teacher when solving a nonroutine geometry problem in a dynamic geometry environment. The main purpose of the study was to uncover and investigate patterns of metacognitive processes and to understand what circumstances, situations, and interactions in a dynamic geometry environment promoted metacognitive behaviors. An adaptation of Schoenfeld's (1981) model of episodes and executive decisions in mathematics problem solving, and the theory of instrumentation (Rabardel, 2001) was used to identify patterns of metacognitive processes in a dynamic geometry environment. During different phases of problem solving the participant engaged in different metacognitive behaviors whereas the dynamic geometry software supported strategies that are available and/or not available on paper and pen. The effectiveness of solution paths was dependent on the presence of managerial decisions, and well-orchestrated usage of different resources, both knowledge and technology. However, the results of the study call to question to which extent engagement in metacognitive behaviors is necessarily desirable or productive.

Keywords: case study, problem solving, metacognition, nonroutine geometry problems, preservice teacher, dynamic geometry software

1 Introduction

At the beginning of the 21st century the rapid mathematization of work in all areas relevant for work and social context influenced the mathematics that society needs. As a consequence, these created unprecedented challenges in school mathematics (Fey, Hollenbeck, & Wray, 2010). Nowadays, topics taught in mathematical classes require more than mere arithmetic or calculation skills, but rather extension and adaptability of previous knowledge, and flexibility in thinking. For that reason, since the 1980s mathematics educators have agreed upon the idea of developing problem solving ability. Nowadays, problem solving plays a prominent role in the curriculum as it is an essential part of mathematical knowledge and performance, and it is considered to be the heart of mathematics (NCTM, 2000; J. W. Wilson, Fernandez, & Hadaway, 1993). Despite the emphasis given to mathematical problem solving by various professional organizations, reform curricula, educators, and researchers, students' performance on challenging problems rarely meets delineated expectations. Research (e.g., Garofalo & Lester, 1985; Schoenfeld, 1985, 1987; Silver, 1994) shows that students' low problem solving performance is due to lack of metacognitive behaviors. That is, behaviors that refer to individual's awareness, consideration, and control of one's cognitive processes (Flavell, 1976). The use of metacognitive processes (e.g., planning, monitoring, testing, revising, and evaluating) supports problem solvers during the solution process and improves their ability to obtain the goal. Thus, metacognition is a critical component in cognitive function and cognitive development.

On the other hand, for some, the appropriate use of technology has become a synonym for problem solving. A plethora of research reported on the importance of technology as a tool for mathematics problem solving, such as different uses of dynamic geometry software (DGS) and heuristic approaches to problem (e.g., Fey et al., 2010; Hollebrands, 2007; NCTM, 2005; J. W. Wilson et al., 1993). Nevertheless, no study addressed the impact of working in dynamic geometry environments on students' metacognitive processes, nor characterized specific strategies related to a particular metacognitive process independently of a context.

With these considerations in mind, the main purpose of the study was to uncover and investigate patterns of metacognitive processes a preservice teacher exhibited when problem solving in a DGS. Moreover, this study's holistic design by integrating both the content (nonroutine problem) and context (technology environment) aimed to seek an understanding about how and why observed metacognitive processes, that is, metacognitive behaviors emerged with respect to the use of a dynamic geometry software use. Central questions of this case study were:

- What are some of the metacognitive processes exhibited by a preservice teacher when engaged in solving nonroutine geometry problem in a dynamic geometry environment?
- What built-in functions of a dynamic geometry software were used by the preservice teacher during problem solving and how was its use associated with exhibited metacognitive processes?

The research results help to extend the current research on student thought processes by understanding better the multi-facet phenomenon of metacognition, and highlight metacognitive processes, and technology use needed for the solver's productive and efficient problem solving effort.

2 Theoretical framework

In the following, I outline three models that allowed me to examine the metacognitive processes during problem solving and helped explain how DGS was integrated into student's problem-solving processes.

2.1 Nature of cognitive processing while problem solving

2.1.1 Cognitive processing while problem solving

Nature of cognitive processing is dual – cognitive and metacognitive. This was pointed out by Flavell (1981) who stated: "We develop cognitive actions or strategies for making cognitive progress and we also develop cognitive actions or strategies for monitoring cognitive progress. The two might be thought of as cognitive strategies and metacognitive strategies" (p. 53). In other words, cognitive processes deal with the information recording and processing. For instance, rehearsal (e.g., underlying, copying), elaboration (e.g., summarizing, paraphrasing), organizational (e.g., making an outline), executional (calculations, drawing a diagram) processes are cognitive processes. Metacognition, on the other hand, has a managerial and a regulatory role of cognitive processes. It focuses our attention on the importance of an individual having management of his or her own thinking and consists of the strategies that he or she uses to plan for their learning, monitor their thinking, and control their thinking (Flavell, 1976). Thus, metacognition is a higher-order system overlooking and managing the cognitive system, but at the same time being part of it.

The line between cognition and metacognition, however, tends to be blurry at time; cognition is implicit in any metacognitive activity, but metacognition might or might not be present during a cognitive act, or may not be apparent or observable (Veenman, Van Hout-Wolters, & Afflerbach 2006). Main reason for this is, that Flavell's definition ignores two key non-regulatory functions of metacognition, namely awareness (individuals have of their own thinking), and evaluation (of these processes). For that reason, in this work I used extension of Flavell's definition by focusing on its three components: awareness, evaluation and regulation (J. Wilson & Clarke, 2004):

- *Metacognitive awareness* refers to "individuals' awareness of where they are in the learning process or in the process of solving a problem, of their content-specific knowledge, and of their knowledge about their personal learning or problem solving strategies" (J. Wilson & Clarke, 2004, p. 27). It entails knowledge of what has been done, what needs to be done, and what might be done in order to attain a specific goal related to learning or a problem-solving situation.
- *Metacognitive evaluation* refers to judgments made with respect to one's thinking processes. For instance, one can evaluate the effectiveness of their thinking or of their strategy choice.
- *Metacognitive regulation* occurs when an individual makes use of his or her metacognitive skills to direct his or her knowledge and thinking. It draws on individuals' knowledge (about self, possessed strategies, and how and why to use them) and uses executive skills (self-correcting, setting goals) to optimize the use of his or hers cognitive resources.

This model of metacognition allowed characterizing each process as being either cognitive or metacognitive (for more details see Sect. 3.4).

2.1.2 Cognitive-metacognitive framework

Various problem-solving models (e.g., Garofalo & Lester, 1985; Pólya, 1945/1973; Schoenfeld, 1981, 1985) describe the thought processes problem solvers use during problem solving. Pólya (1945/1973) suggests a well-known model of problem solving that consists of four phases that are held to be essential in problem solving when applied to a mathematics problem. In the following I outline Pólya's model:

- *Understanding the problem*. Understand the verbal statement of the problem. Determine the unknown, data and condition. Analyze the data and condition. Draw a figure and introduce suitable notation. Separate specific parts of the condition for better understanding.
- *Devising a plan*. Try to find a connection between the data and the problem. Consider whether the earlier methods or auxiliary problems can be used now. Develop a plan considering which calculations, computations and/or construction to perform in order to obtain the unknown.
- *Carrying out the plan*. Examine the solution plan. Check each step of the plan carefully to make sure that each step is correct. Implement the solution plan.
- *Looking back*. Reexamine and reconsider the solution by checking the result and checking the argument. Examine the reasonableness of the solution. Consider deriving the result differently, using the result, or the method or some other problem, or stating a new problem to solve.

Of course, this process is not linear, but rather dynamic, cyclic, and iterative in its nature as illustrated in Figure 1 (J. W. Wilson et al., 1993, p. 61). False moves may occur, and the problem solver needs to simultaneously monitor his or her progress and go back to previous moves again and again, and change strategies, if necessary, until the goal is reached. For example, in the attempt of making a good plan the student may discover a need to understand the problem better; the student then has to go back to develop a new understanding of the problem. In this problem-solving model, any of the arrows can describe students' thinking processes during mathematics problem-solving activity.

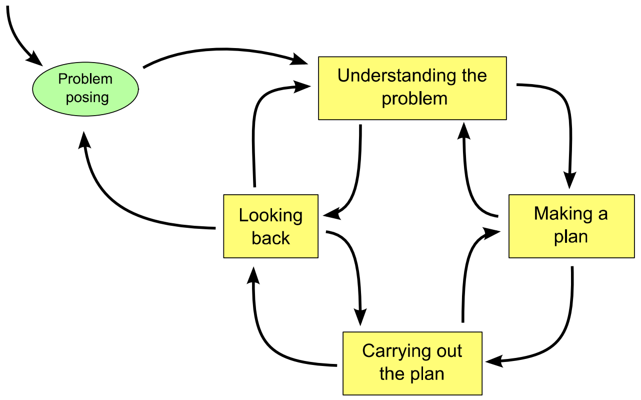


Figure 1. Dynamic and cyclic interpretation of Pólya's model by J. W. Wilson et al. (1993). Note: Reconstructed from „Mathematical problem solving," by J. W. Wilson et al., p. 61, Research ideas for the classroom: High school mathematics, New York, Macmillan.

Whereas frameworks by Garofalo and Lester (1985) and Pólya (1945/1973), outline distinct phases of activity, Schoenfeld's (1981) framework focuses on problem-solving behaviors (cognitive and metacognitive actions) without creating clusters of categories. Problem-solving protocols (transcripts) are parsed into episodes: reading, analysis, exploration, planning/implementation, and verification, together with junctions between episodes (transitions). These episodes present periods of time during which a problem solver is engaged in a particular activity, such as exploring different possibilities or planning the best solution. Decision-making behaviors are analyzed by examining each episode and the transition between them using a set of predetermined questions attributed to each episode or transition.

Adapting the models by Pólya (1945/1973), Schoenfeld (1981, 1985), and Garofalo and Lester (1985), I developed a model capturing together student's cognitive and metacognitive behaviors when problem solving in DGS (Kuzle, 2011) and their complex interplay. On the contrary to Schoenfeld (1981, 1985) the planning/implementation episodes were separated into two episodes, as they do not necessarily occur simultaneously or sequentially. In addition understanding episode as suggested by Pólya was added to report on behaviors important in the problem solving process. The resulting model was characterized by the following episodes: (1) reading the problem, (2) understanding the problem, (3) analyzing what needs to be done, (4) exploring different possibilities, (5) planning the best solution, (6) implementing the plan, and (7) verifying the answer is a solution, together with junctions between episodes (transition). In the following I shorty describe each of the episodes with some possible problem solving behaviors. In addition, figure 2 offers an illustration of this model.

In a *reading episode*, student reads the problem statement silently or aloud, whereas then in an *understanding episode* he or she may note conditions of the problem, state the goals of the problem, represent the problem, and assess his or her current knowledge relative to the task. In the *analysis episode*, the student attempts to understand the problem, decompose the problem in its basic elements, examine the relationships between the given information, conditions and the goals of the problem, and choose appropriate perspectives to solve the problem using different strategies to assist him or her (e.g., draw diagram if at all possible, examine special cases, simplify problem). Whereas an analysis episode is well-structured, an exploration episode is less structured and removed from the given problem. In an *exploration episode*, the student searches for relevant information that can be used in an analysis, planning, and implementation sequence relying on his or her previous knowledge and experience. In a *planning/implementation* episode, the student creates a plan and implements it. In a *verification episode*, the student reviews and tests whether his or her solution passes specific or general tests in relation to requirements of the problem. A *transition episode* is a junction between the other episodes and occurs only when a student assesses the current solution state and makes decisions about pursuing a new direction to solve the problem.

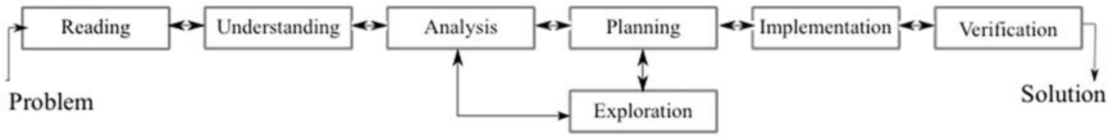


Figure 2. Cognitive-metacognitive framework by Kuzle (2011, 2013).

2.2 Instrumental approach to technology use

To explain student-tool interaction during problem solving, I drew on the theory of instrumentation developed in cognitive ergonomics, which is part of instrumental approach (Rabardel, 2001). Though this theoretical frame is very general dealing with different types of tools, it has been proven fruitful when examining students' mathematical activities and understanding their difficulties of effective use of technology (CAS, DGS), within the instrumentation perspective (e.g., Artigue, 2002; Kieran & Drijvers, 2006). Two aspects from these studies need to be highlighted: the instrumental approach and framing schemes. The main tenet in the instrumental approach is the difference drawn between an artifact and an instrument. A technical system does not immediately constitute a tool for the user; it becomes an instrument when the subject has been able to appropriate it for himself. In other words, the artifact is the object that is used as a tool, whereas the instrument involves techniques and schemes developed by the user during its use (instrumentalisation) that then guide the way the tool is being used and the user's thinking (instrumentation). *Instrumentalisation* is a psychological process that refers to the construction of schemes[1] by the user directed towards the technological tool; that is, it refers to the appropriation and transformation of the tool by the student. This process can lead to enrichment of an artifact, or to its impoverishment. For instance instrumentalisation processes includes behaviors such as user coordinating the use of both geometric and algebraic windows and catalyze at the same time geometrical understanding. *Instrumentation* is directed towards mathematical concepts; that is, it refers to the effects of tool use on the student activity, where the use of a certain tool becomes internalized as the user executes the task. For instance, understanding and applying DGS tools for solving the problems or creating new tools are some of the possible instrumentation processes a user may use.

When an artifact becomes a meaningful instrument through the process of instrumentalisation and instrumentation, then this process is called *instrumental genesis*. Thus, the idea of instrumental genesis reflects the fact that using a tool is not a one-way process; there is dialectic between the subject acting on his/her personal instrument and the instrument acting on the subject's thinking. This process is complex and depends on the characteristics of the artifact, its constraints and affordances, and also on the knowledge of the user. I used this theory to better understand the questions of technological integration

[1] A scheme is a stable mental organization, which includes both technical skills and supporting concepts for a way of using the artifact for a given class of tasks.

during problem solving, that is, how the tool shapes the thinking of the user, but also how it is shaped by the user's thinking.

3 Methods: Context, data collection and analysis

3.1 Case study research design and participant selection

For this study, a case study qualitative research design was chosen which allows naturalistic generalization that develop with a person as a result of an experience. That said, examining one case provides insight into areas that is relatively unknown – how a preservice teacher might use DGS during problem solving and how its usage interacts with cognitive-metacognitive enterprise – but worth reporting on. Based on both research and personal experience, one participant was determined – Wes – that would be ideal; not only he had been used to working in a DGS, but worked well individually, was a reflective thinker, and had substantial mathematical background.

Wes was a senior student in the mathematics education department at the time of data collection. He had substantial problem solving experience: he took both a problem-solving course at the university, and had gathered problem solving experience in school. The problem solving course gave him the opportunity to solve many problems, and to experience genuine problem solving.

3.2 Data collection procedures

Data collection methods for this study consisted of the following: (1) interviews, (2) audio and video-data from the think-aloud sessions, (3) document review (of participant's written solutions, DGS files), and (4) the researcher's observations of the sessions during problem solving in a DGS. The first phase of formal data collection started by using the preliminary interview protocol that was intended to elicit the participant's background. Following the preliminary interview, the participant shortly practiced the think-aloud protocol with a sample problem. The procedure provided him with important practice for understanding and developing confidence prior to utilizing the technique with the research problems. At the core of the present study, the next stage of data collection concentrated on the participant's involvement in investigations of three mathematical problems in DGS. Data collection occurred in a one-to-one setting between the participant and the author. At the beginning of each session I provided the participant with a sheet of paper and a pen as well as opened the DGS file on which the given problem was printed. The participant continuously thought aloud and engaged in a conversation with the author while working on the problems describing his thinking and behaviors. However, during extended periods of time I used the following prompts to encourage the participant to speak his thoughts: "keep explaining aloud what you are thinking," "say everything you are thinking and doing," "say everything in your mind," or "tell me how are you using technology in this situation." He used as much time as needed in solving each problem. The individual interviews took place shortly after the participant finished solving each problem where we talked comfortably about the participant's problem solving session: (1) participant's views about the problem solving task and (2) problem solving process, and (3) how these interacted with his use of technology. The same procedures were used for the following two mathematical tasks. An additional data resource was my own field notes, which included the descriptions of questions, reactions, and behaviors that occurred during data collection that were then used during retrospective interview.

3.3 The mathematical task

Wes solved three different problems, that were chosen such that they demanded strategy flexibility, thinking flexibility, provided participants with opportunities to engage in metacognitive activity, and covered mathematical content area in geometry. For the purpose of the paper, I report on Wes's solution of the *Airport Problem* (see below). This was an exploration problem, and it is used here because it provided rich exhibition of metacognitive behaviors, and different uses of the DGS. The Airport problem is an open form of *Viviani's Theorem* that was chosen for the following criteria: relevancy to the participant's experience, supporting experimentation, conjecturing, rejecting conjectures and participant's capacity to solve it. For instance, to find a possible location for the airport, one drags the point Airport discovering a striking finding: Any point inside the triangle ABC satisfies the problem criterion.

The airport problem.

Three towns, Athens, Bogart and Columbus, are equally distant from each other and connected by straight roads. An airport will be constructed such that the sum of its distances to the roads is as small as possible. What are possible locations for the airport?

Justify your answers as best as you can.

3.4 Data analysis, validity and reliability, and coding scheme

For the purpose of this study, multiple stages of analysis, as suggested by Patton (2002) were conducted using a deductive approach, which is explained in the following two paragraphs.

The process of analyzing the data started with selecting relevant parts of the problem solving session for intensive examination, which were then transcribed (think-aloud session (TA), retrospective interview (RI), final interview (FI)). The transcripts were analyzed following Kuzle's (2011, 2013) cognitive-metacognitive framework, which involved identifying 7 types of macroscopic episodes, and the transitions between the episodes. I looked for details (characteristic behaviors), which were first noted, as outlined in the framework. The behaviors needed to be explicit; otherwise, they were not coded. After the behaviors were identified, these wire clustered in an "episode" cluster on the basis of episode descriptors (see Sect. 2.1.1). In addition, I noted if any behaviors was prompted by DGS and vice versa. In order to decide on the cognitive level of each episode, the problem solving processes were analyzed and categorized as either cognitive or metacognitive. This was particularly difficult to code. It was difficult to distinguish between a participant thinking or encouraging himself to think and thinking about his thinking—that is, to be metacognitive. In addition, it was sometimes difficult to discern if during a cognitive act metacognition was or was not present, or was simply not observable. Here the three metacognitive constructs – awareness, evaluation, and regulation – were considered (J. Wilson & Clarke, 2004) to help make a decision. If any of these three constructs was present, the behavior was coded as metacognitive. Transcripts from the retrospective interview were here particularly helpful, because often the participant gave explanations for his actions. This process was used to answer the first research question.

A posteriori, instrumentation theory (Rabardel, 2001) guided the data analysis with respect to the second research question. I first noted what different DGS affordances were used within each episode. These included drawing objects, figures and auxiliary lines, constructing objects, using editing, transformational and measurement tools, calculations, dragging, tracing and locus. In additional step, I looked at how the availability of DGS influenced the decisions participants made with respect to solving the problem, the role that technology played in the mathematical work, and the level of thinking needed for students to appropriately use the technology when problem solving. On the basis of these considerations, a behavior was coded as instrumentation or instrumentalisation. Table 1 offers an overview of a coding scheme from the coding-manual used during the data analysis. The first letter denotes the letter of the category (resource, method, type of episode), whereas the following letters denote different accompanying subcategories. Table 2 offers a short extract from an interview with codes. In the first column, I offer an excerpt from two different episodes, complementing think-aloud session with retrospective interview and/or final interview. The latter offered support when deciding about the nature of cognitive processing, which is presented in the second and third column, retrospectively.

Table 1. Overview of a coding scheme

Resources		
	RMK	Mathematical knowledge, facts, and procedures
	RT	Technology
Methods		
	MCI	Constructs new ideas or statements
	MAR	Accesses resources
	MCP	Carries our a procedure (e.g., computation)
Understanding episode		
	UEUP	Effort is put to understand the problem (sense making)
	UESG	Stating the goals of the problem
	UMS	Monitoring strategy
	UERP	Representing the problem
	UEAK	Assessing current mathematical knowledge relative to the task
	UES	Considering, devising, and selecting strategies and tools
	…	…

Table 2. Wes's transcript excertps from an interview with codes

Excerpt	Metacognitive behaviors	Code
TA: (Reads aloud problem statements with pauses between each one of them and from the hard copy.) RI: Because sometimes if you read through a problem quickly you miss a piece of information or you will misread something and then it doesn't matter what you do afterwards because it's gonna be wrong.	Monitoring strategy Control strategy	RMS RCS
TA: What are the possible locations? What's the best location for the airport? That's more of a practical question because you are you have to think about the context of the problem. You have three towns and you have an airport, you want to minimize the distance from each one and you want it to be an equal distance so that it's just an efficient location.	Internal dialogue - monitoring strategy Evaluation - judging the effectiveness of thinking processes and strategy Awareness and regulation - mathematical knowledge	UMS UEEP UEAK URMK

To assure validity and reliability of the study, I used triangulation of sources and analyst triangulation. Triangulation of qualitative data sources involved comparing observations with interviews and checking for the consistency of what the participant said during the think-aloud session and during the interview session. Finally with regard to analyst triangulation, I used member checking and another expert. The participant read the analysis of the session, after which we met in person and discussed any questionable interpretation. With respect to the latter, another expert in the area of problem solving received the framework as well as a coding manual that was developed by assigning examples of answers to each subcategory (episodes and behaviors representative to that category and its nature, user-technology interaction with descriptions). In addition, when we were doubtful of assigning a category and/or nature of cognitive processing, we watched together the session part in questions and read related transcripts, and discussed these until an agreement was met. The employment of the procedures mentioned above ensured trustworthiness and rigor.

4 Results

This paper presents findings from a case study related to a secondary mathematics preservice teacher. Within this case study, one teacher is in the focus: Wes. The following section provides brief synthesis of Wes's session followed by a detailed representation and analysis of Wes's one problem solving session, his use of technology and the factors that fostered or hindered his metacognitive processes.

4.1 Solving the airport problem with a focus on metacognitive processes

4.1.1 Synthesis of Wes's problem solving space

Wes started the problem solving session by reading the problem. In order to get a better understanding of the problem, he made a static representation of the problem. During that time he accessed and selected the knowledge relevant to the task. Already at this early stage he conjectured that points of concurrency are a plausible result of the problem. He then moved onto the computer where he made a dynamic representation of the problem. Often he reminded himself of the conditions and requirements of the problem. Through this process he developed a deep understanding of the problem relating together both mathematical content and context. Because of the exploratory nature of the problem, he decided to use the dynamic capabilities of the software in his search for a solution plan; make and test his conjecture. During this process, allocation of resources, namely content-specific knowledge and the software was prominent. The nature of student-tool use led him to refuting his conjecture, and making a new one was a result of DGS use. He verified his conjecture by using DGS dragging modalities with which the session ended.

4.1.2 Wes's metacognitive processes when engaged in problem solving in a DGS

Here I outline Wes's exhibited metacognitive behaviors on the basis of the outlined cognitive-metacognitive framework (see Sect. 2.1.1). Nevertheless, I also integrate the use of DGS during problem. In each episode one or more excerpts of events is given which is then followed by their interpretation. In the Sect. 4.2 I focus in more details on the use of built-in functions and its association with either prompting or inducing different metacognitive behaviors.

Reading episode. Wes started the session by reading the problem statement aloud and from the hard copy (see Table 3 for details).

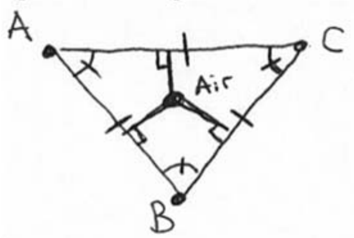
Table 3. Reading episode

	Excerpt	**Metacognitive behaviors**
Wes	Three towns, Athens, Bogart and Columbus, are equally distant from each other and connected by straight roads. An airport will be constructed such that the sum of its distances to the roads is as small as possible. What are possible locations for the airport? Justify your answers as best as you can.	
Interviewer	How come you read the problem from the hard copy?	
Wes	If it's not something I am familiar with, I like having hard copies. There is something about it. I like to have hard copies.	Monitoring strategy that help maintain focus and identify the problem components
Wes	(Reads the problem again).	Control strategy to avoid missteps

The reading episode was not led by cognitive action of reading itself; Wes engaged in two different metacognitive behaviors as outline above. Engagement in these monitoring and control strategies was a metacognitive behavior where he drew upon his previous problem solving experience. Acting on these metacognitive processes prompted metacognitive behaviors aligned with the understanding episode that contributed moving through the problem solving space.

Understanding episode. After having read the problem, he started representing the problem on the problem sheet by first noting the givens and goal of the problem, and assessed his current knowledge relative to the task. He then moved onto the DGS repeating his actions (see Table 4 for details).

Table 4. Understainding episode

	Excerpt	Metacognitive behaviors
Wes	It says that the three towns are equally distant from one other so C has to be equidistant from B and from A and A has to be equally distant from B and C. So I figured it would be an equilateral triangle.	Sense making - engaging with the problem text, interpreting the problem Accessing relevant knowledge
		
	(Sequential steps of reading sentence part and drawing diagram).	Control strategy - diving the problem into subproblems
	I just put that point in the middle, there is no exact location at this point yet since I am thinking that the distances need to be perpendicular distances because the distance from a point to a line should be the smallest distance possible which is a perpendicular line dropped from there [Point Air].	Regulation - deciding to place a free point outside the triangle Orchestration of resources - accessing and considering relevant mathematical knowledge on the basis of the drawn diagram
	Did I cover it all? Hm...yeees.	Control strategy - internal dialogue
	(Restating the problem in own words.)	Monitoring strategy - reengaging with the problem text
	Well if it's an equilateral triangle all the points of the concurrency are in the middle. I was getting a rough idea in my head. I like to when I break a question at its parts to have some kind of visual representation so I can as I go along add things to the representation and eventually in the end I have a visual representation of what the question is asking me.	Awareness - considering and accessing previous content specific knowledge and experiences
	Now's the time to use this. Now I have an idea what's going on.	Regulation - effort put forth to fully comprehend the problem, and allocate knowledge that might be helpful before trying to solve it and use technology
	I need to construct an equilateral triangle. I will draw a segment, and I will label that Segment AB, and then I will take two circles, one centered at A to B, and one centered from B to A. So I can conclude this since this radius [AB] is also the same as this radius [AC], and this radius [BA] is same to this radius [BC], by the transitive property I have an equilateral triangle. So this intersection point then is Point C, which is Columbus in the problem.	Instrumentalization - delegating constructing work to DGS
	I am gonna put it out here just because I have an idea it might be in there but it's not necessarily true. It might be able to be outside as well. Because the way I am gonna construct this is to move that move that point wherever I need it to be.	Regulation & instrumentalization - selecting a random point that can be dragged and constructing perpendiculars (taking into account DGS affordances)

In summary, Wes spent a great amount of time engaging in variety of problem solving behaviors during the understanding episode. As noted earlier, he looked for the given information in the problem and what he was asked for, restated the problem, reengaged with the problem text, made sense of the problem information, represented the givens and the goals of the problem by making a representation of the problem, introduced a suitable notation, and reminded himself of the requirements of the problem. These monitoring strategies were metacognitive behaviors that were an important attribute during problem solving that helped not only develop an understanding of the problem, and access his knowledge and strategies but contributed to moving to next problem solving episode. After he stated the conjecture as a result of his current problem solving stage, and through the engagement in metacognitive behaviors (e.g., visualizing, accessing, considering, and manipulating mathematical knowledge, concepts, and facts relevant to the problem), he transitioned to the planning episode.

Planning and implementation episode. In the next step he engaged in devising a solution plan for the problem relying on his previous content-specific knowledge. He decided to take a free point and test it against his Airport point – circumcenter (see Table 5 for details).

Table 5. Planning and implementation episode

	Excerpt	**Metacognitive behaviors**
Wes	I am gonna make a simulation of the problem on the GSP[2], make a dynamic point and use that point to test the conjecture of the circumcenter.	Instrumentalisation - dragging
	And now what I am gonna do is, I am going to calculate the sum of those because that's what it asks. It says an airport will be constructed such that the sum of its distances to the roads.	Instrumentalisation - measuring
	My original expectation was gonna be the centroid of the triangle for an airport will be constructed such that the sum of its distances to the roads is small as possible. I think that's where it will be.	Awareness - accessing and considering mathematical concepts and facts that might be useful

Verification episode. At this point Wes was surprised with his discovery, and for that reason revised his conjectured and tested it with respect to the problem requirements (see Table 6 for details).

2 GSP stands for the Geometer's Sketchpad. GSP is a DGS, which was used in the study.

Table 6. Verification episode

	Excerpt	Metacognitive behaviors
Wes	(Moves the Airport Point in the plane observing how the distances change). 6.84 [cm] and it's constant inside of here and when we go outside it starts to change. It starts to increase and then it goes back to a constant [inside] and it starts to increase [outside].	Instrumental genesis - testing, rejecting, revising and refining conjecture on the basis of visual input
	Well, I think I know now what the possible locations are. I would say the possible locations anywhere inside the equilateral triangle.	Instrumentation - conjecturing as a result of visualization

Though, at first his actions influenced the use of the software, feedback that it provides to the user then shaped the use of the software as an exploration tool that continued to influence his actions. With this implementation and verification episode ended, and the problem was solved although he was quite perplexed by the result.

4.2 Use of DGS built-in functions and their association with the exhibited metacognitive processes

The DGS appeared to be integrated into the problem solving processes and strategies used by Wes. His knowledge base of the DGS capabilities affected the extent to which he used the DGS while solving the Airport problem. He exhibited knowledge of several built-in functions of the software. The CONSTRUCT and MEASURE menus were used during understanding, analysis, exploring, implementing and verifying episodes. The CONSTRUCT functions allowed him to make a representation of the problem and quickly add secondary elements, lines, segments, rays, and points in the figure. Hence, the use of DGS supported making a representation of the problem which contributed to visaulization of the problem and developing an understanding of the problem. Interpreting the problem and directing his knowledge relevant to the problem was important to successfully represent the problem where he stayed mentally engaged throughout the process monitoring his thinking. Visualization of the problem triggered organizing his knowledge when seeking relationships between the conditions and the goals of the problem. These metacognitive and cognitive processes often directed his actions and thinking into understanding the information obtained through the use of DGS, followed by accessing, considering and eventually selecting a perspective as well as allowing monitoring progress of undertaken activities. The MEASURE functions allowed the participant to formulate, test, and verify a conjecture or a plan. Wes often used the Measurement Tool to verify his thinking and then as a result of that action to decide on a strategy going forward. Hence, the appropriation of the software to a measurement and verifying tool was influnced by his thinking processes.

Not having a static but a dynamic representation of the problem drove the use of DGS as an exploration tool. The dynamic character of the software allowed Wes to manipulate the figure or parts of the figure by DRAGGING function. For instance, dragging is a cognitive behavior that is not available on paper-and-pen, that is, in non-dynamic environments. Dragging as a strategy was not randomly chosen, but used to develop a better understanding of the current problem solving situation, gather information, help formulate a conjecture or a plan and assess different ideas or actions. That is, choice of dragging as a strategy for the above named purpose was a metacognitive behavior where they drew on their previous knowledge of how and why to use the particular strategy and by using their executive skill to optimize the use of their resources. DGS helped develop his acitivities; he transformed DGS into an exploration tool to fit his needs taking into consideration its affordances and constraints throughout their problem solving space. The nature of the software allowed him to assess their solution state and influenced his thinking and further actions. He was able to make sense of the problem solving situtation, evaluate his thinking processes, correct them and direct their thinking processes and actions as a result of previous actions towards achieving their current goals. It was a metacognitive act that dealt with reflecting on the task and undertaken activities as a result of feedback provided by the DGS in directing their thinking processes towards a solution of the task, and choosing of the strategy when working through problem solving space.

To sum up, the interplay between thinking processes and tool use was a prominent problem solving behavior. The problem solving context allowed Wes to use and apply his knowledge, translate verbal statements into an interactive representation, investigate a mathematical idea, deal with a situation that may not have a single solution, and make, test and verify his conjectures. Thus, the nature of the software supported exhibiting different mathematical thinking processes, the use of different strategies and different uses of the DGS. Through it a variety of cognitive behaviors became available or were supported by the dynamic environment. Last but not least, exhibited metacognitive processes were tied to his use of DGS during problem solving, but his ability to decide how, when, and whether to use it determined the extent to which his efforts were productive or not. In Table 7 a summary of the both subsections is presented as it relates to the research questions.

Table 7. Summary of results with respect to the metacognitive behaviors and their association with the technology use

Episode	Metacognitive behavior	Technology use
Reading	Monitoring strategy Control strategy	
Understanding	Awareness - deciding to draw a diagram to visualize the problem and attain ideas for problem solving approach Monitoring - restating the problem, reengaging with the problem text, interpreting the problem Regulation - deciding to place a free point outside the triangle in order to be able to check all solutions Awareness - accessing and considering relevant mathematical knowledge on the basis of the drawn diagram that is potentially related to problem situation Awareness - considering and devising strategies and tools that are potentially related to problem situation Control strategy - internal dialogue, dividing the problem into subproblems)	CONSTRUCT diagram and auxiliary lines VISUALIZATION - accurate visual representation CONSTRUCT free points
Planning	Regulation - orchestration of various resources Awareness - devising and selecting steps and strategy for solving the problem that might potentially lead to problem solution on the basis of mathematical knowledge	Strategy was chosen as a result of DGS exploratory nature
Implementation	Awareness - executing strategy Reflects and evaluates undertaken activities Reflecting on the problem goal	CONSTRUCT MEASURE
Verification	Monitoring - engagement in internal dialogue Evaluation of strategy choice through testing, rejecting, revising and refining conjecture Evaluating the reasonableness of the solution	DRAGGING modalities VISUAL FEEDBACK as a result of dragging modalities Self-initiated EXPLORATIONS

5 Discussion

All of the episodes as outlined in the compilation of the cognitive-metacognitive model were identified in this study. With respect to metacognitive processes within each of the episodes, it was evident that awareness of one’s knowledge triggered selective attention, evaluation of one’s thinking helped better planning for effective solution approaches, and regulation of one’s thinking helped monitor progress, select appropriate problem-solving strategies, and regulate missteps. Hence, these skills proved to be important for productive problem solving activity, which was also noted earlier by Schoenfeld (1987). Similarly to previous research (Carlson & Bloom, 2005; J. Wilson & Clarke, 2004), it appeared that a continuous interplay between cognitive, and metacognitive behaviors and strategies was paramount for successful, and productive problem solving. Problem solvers develop cognitive actions and strategies to make cognitive progress, while at the same time these are important to monitor cognitive processes (Flavell, 1981). More closely, behaviors exhibited by the participant provided a detailed characterization of the interplay between metacognitive processes and conceptual knowledge that influenced most of the episodes of the problem-solving process.

Even though the work of Schoenfeld (1981, 1985) in which he examined problem solving of undergraduate students gave me a solid basis to look at metacognitive processes, this study extended the work of Schoenfeld by examining the patterns of cognitive and metacognitive behaviors in a different problem-solving context, that is, in DGS. I have observed that technology seems to permit the problem solver to focus on the larger overarching concept and make connections that would otherwise be lost if technology was not available. Wes used the software's capabilities, precision, measuring, dragging, and construction to engage in problem solving activities which proved to be a cyclic process of generation, justification, and refinement of a plausible solution experiencing genuine problem solving. Thus, a variety of both cognitive and metacognitive behaviors are supported or easier to occur by the DGS. Continuos interplay of cognitive and metacognitive behaviors together with well-though-out use of the DGS was important for productive problem solving. For instance, DGS affordances helped explore, gather information, experiment, conjecture, better understand the problem, remember mathematical concepts, attain accurate visual input and trigger possible solution possibilities. Wes instrumentalised such behaviors; he drew on his previous knowledge of how and why to use the particular strategy and by using his executive skill to optimize the use of available resources. Nevertheless, the tool itself influenced his way of thinking and behaviors. The feedback provided by the DGS through dragging was critical for the participant's later decisions and actions as suggested by other researchers (Hollebrands, 2007; Olive & Makar, 2010; J. W. Wilson et al., 1993). Hence, DGS proved to be an important resource when working on nonroutine problems benefiting the problem solving process; it supported participant to engage in processes, such as gaining insight, pattern recognition, discovering, exploring, conjecturing, abstracting, and supported flexibility in thinking, transfer of mathematical knowledge to unfamiliar situations and extension of previous knowledge and concepts as reported by Zbiek et al. (2007) and Kuzle (2013, 2015). Exhibited metacognitive processes were tied to the problem solver's use of the DGS during problem solving, but his ability to decide how, when, and whether to use it determined the extent to which his efforts were effective, and efficient.

There is no doubt that the participant in this study engaged in metacognitive acts. However, the data from the study demonstrated that frequent use of metacognitive acts does not always equate with productive problem-solving activities. For instance, Wes spent an extensive period of time exploring if centroid was the solution to the Airport Problem, and took him quite some time to realize approach was unproductive. Instead, he could have started with an arbitrary point, which would have been more productive. Schoenfeld (1992) argued that we are still missing an adequate theoretical model that would explain the mechanisms of metacognition. For instance, negative effects of metacognitive processes such as metacognitive awareness or regulation of one's cognitive processes can hinder participants' problem solving efforts as was observed in this study and in the study by DeFranco (1996) and Goos (2002). Teachers and educators may believe that metacognitive behaviors are important during problem solving activities and that these behaviors should be for that reason reinforced in their instruction, however, the results from the study raised important concerns with respect to what extent are metacognitive behaviors desirable or productive. Further research on metacognition focusing on exploring possible unproductive aspects of metacognitive activity while problem solving is crucial. Results of such research could contribute to an improved theory of metacognition.

The implementation of DGS in the classroom affects not only pedagogy, but student cognition as well (Kuzle, 2015). In addition, teachers themselves lack an understanding of the complex and multi-faced phenomenon of metacognition (Veenman et al., 2006). Hence, preservice teachers before becoming inservice teachers and taking those responsibilities on themselves should have experience in genuine technology problem solving as well as opportunities to discuss curricular, pedagogical and learning issues with respect to that mission, and metacognitive aspects of problem solving in variety of contexts. Characterization of preservice teachers' metacognitive processes may help educators effectively plan, develop and adjust preservice teacher programs to support their development.

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