Design and evaluation of practice-oriented materials fostering students' development of problem-solving competence: The case of working backward strategy

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In a design-research project on problem-solving, theory-based and practice-oriented materials were developed with the goal of fostering systematical development of students' problem-solving competence in a targeted manner by learning heuristics. Special attention was given to working backward strategy, which has been shown difficult for students to learn and use. In the study, 14 Grade 5 students participated in explicit heuristic training. The results show that even though the students intuitively reversed their thought processes before the explicit training, they experienced difficulties when solving complex reversing tasks, which improved considerably after explicit heuristic training. Thus, the study results showed that the developed materials using design-based research-approach promoted the development of students' flexibility of thought when problem-solving by working backward. At the end of the paper, the results are discussed with regard to their theoretical and practical implications.

Keywords
design research,
mathematics education,
problem-solving,
working backward strategy,
reversibility,
explicit heuristic training

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1 Introduction

Problem-solving is a binding process standard in different educational systems (e.g., Finnish National Board of Education [FNBE], 2004, 2014; National Council of Teachers of Mathematics [NCTM], 2000; The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany [KMK], 2005) that is often neglected in school mathematics (e.g., Gebel & Kuzle, 2019). The plethora of research on problem-solving undergoing since the 1970s identified pivotal practices for problem-solving instruction (e.g., Grouws, 2003; Kilpatrick, 1985; Lester, 1985). Despite more than five decades of this accumulated knowledge, both empirical studies, as well as large-scale studies (e.g., PISA, TIMS study), reported that students are often unable to solve problem tasks. Moreover, teachers lack practical teaching materials to foster students' development of problem-solving competence and at the same time to consolidate their competence in the area (Gebel, 2015; Gebel & Kuzle, 2019; Kuzle & Gebel, 2016). In the context of this reform agenda, collaborative work between educational researchers and practitioners working on issues of everyday practice is crucial in order to overcome the gap between theory and practice (Jahn, 2014). Design-based research (DBR) as a research paradigm that brings the two poles – theory and practice – together, may help overcome this gap (Wang & Hannafin, 2005). In order to close the above-described problem-solving gap, theory-based and practice-oriented materials for middle-grade students were developed in the SymPa project (Systematical and material-based development of problem-solving abilities) in accordance with DBR methodology (Gebel, 2015; Kuzle, 2017a, 2017b; Kuzle & Gebel, 2016). The goal of the SymPa project was to promote the development of problem-solving abilities in Grades 4-6 (Gebel, 2015; Kuzle, 2017a, 2017b; Kuzle & Gebel, 2016).

In design research, depending on the existing object of investigation and the associated restrictions, different design aspects can be focused on that consequently allow to understand possible connections with regard to the fulfilment of the function of the design in a multifaceted way (Jahn, 2014). According to Collins, Joseph, and Bielaczyc (2004), different aspects are relevant for the multi-perspective educational design analysis: cognitive level, interpersonal level, group or classroom level, resource level, and institutional or school level. During the first seven DBR cycles within the SymPa project, the project evaluation focused on developing suitable and sustainable problem-solving materials for their implementation in practice (Kuzle, 2017b; Kuzle & Gebel, 2016) as well as on identifying the design elements contributing to the improvement of the problem-solving competence (Kuzle, 2017a). In other words, the resource level of the educational design was in the foreground of the analysis (Collins et al., 2004). At the same time, the project was analyzed with respect to factors, and conditions that favored and hindered the implementation of the materials in practice on the basis of two DBR cycles (Kuzle, 2017b). Thus, the institutional level of the educational design was analyzed (Collins et al., 2004). Hence, the first phase of the project had more practical output within educational design research.

Relating to the motive of enhancing the quality of research findings, the focus of this paper lies on another aspect relevant to educational design research, namely on the cognitive level (Collins et al., 2004). Specifically, the question about how the design of theory-based and practice-oriented materials for systematical development of mathematical problem-solving competence as well as on how explicit heuristic training organized around these materials affect the thinking and learning of participants over time, and subsequently their increase of knowledge in the context of mathematical problem-solving. This is exemplarily shown with respect to the strategy of working backward, which has been shown difficult for students to learn and use (Aßmus, 2010a, 2010b), albeit its potential in mathematics lessons and importance in everyday life. Through reversible thinking, an individual is capable of seeing things not only from one single perspective but also its reversal. It may also minimize both errors in every decision as well as the error of answers as students tend to review their answers by reversing the result to the initial value of the problem. Lastly, thinking reversible is one of the primary requirements to solve mathematical problems (Bruder & Collet, 2011; Krutetskii, 1976; Lompscher, 1975).

In the following sections, I outline relevant theoretical and methodological underpinnings for systematical development of problem-solving competence in the context of working backward, before showing how these got integrated into students' problem-solving material. On the basis of the educational design research, the development of students' ability to use the strategy of working backward when problem solving is presented. In the last section, I discuss the findings with regard to their theoretical and practical implications.

2 Theoretical foundation

2.1 Mental agility

Schoenfeld (1985) defined the concept of a problem as a subjective assessment:

> The difficulty with defining the term problem is that problem solving is relative. The same tasks that call for significant effort from some students may well be routine exercises for others, and answering them may just be a matter of recall for a given mathematician. Thus, being a 'problem' is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. (p. 75)

Similarly, Bruder and Collet (2011) define the concept of the problem as person-dependent. For them, a task becomes a problem for an individual when it seems unfamiliar, and when a promising solution is not immediately at hand. In that manner, problem-solving refers to a directed cognitive process in which the problem solver determines how to overcome an individual barrier resulting from bringing the initial state to the target state (Bruder & Collet, 2011; Schunk, 2008).

Problem-solving competence relates to cognitive (here heuristic), motivational and volitional knowledge, skills and actions of an individual required for independent and effective dealing with mathematical problems (Bruder & Collet, 2011). Accordingly, each individual must develop the ability to solve problems independently (e.g., Gebel & Kuzle, 2019; Stanic & Kilpatrick, 1989), and should learn approaches (heuristics) for solving mathematical problems and how to apply them in a given situation, develop reflectivity on own actions, and develop willingness to work hard (KMK, 2005; NCTM, 2000).

Research (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985) showed that problem-solving activities in mathematics require skills and understanding that are often not readily apparent to novice problem solvers compared to experienced problem solvers. Especially, intuitive problem solvers possess particular mental agility (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016), which is fundamental to mathematical problem-solving. Lompscher (1975) defined the concept of *mental agility* as a performance characteristic of the individual, which provides both quantitative as well as qualitative characteristics, which influence the intellectual activity. By this, Lompscher (1975) understands the ability to analyze the objective reality of the subject. Consequently, mental agility develops from this very activity which the subject exercises in the mental process in interrelation with objective reality. Accordingly, the mental mastery of the performance requirements of objective reality is expressed through the abilities of the subject. Lompscher (1975) described the mental agility through three subdomains. First of all, the mental operation is considered, which contains solidified action sequences. Processed knowledge forms networks in the long-term memory and, when applied, characterizes its quality (course quality). The aforementioned forms the second subdomain of mental agility and is supplemented by the willingness to actively apply one's knowledge (attitude) (Lompscher, 1975). Here, I mainly limit myself to the first two subdomains.

Mental operations concretize every cognitive activity. Regardless of the object of the action – the goal or the content of the action – they form complex sequences of operations. This results in mental operations, among other things, during the examination of things and characteristics as well as in problem-solving situations (Lompscher, 1975). In addition, every mental action is characterized by content (e.g., concepts, connections, procedures), process (e.g., systematic planning, independence, accuracy, agility), and partially conscious goals and motives. One of the most important mathematically relevant progression qualities is mental agility. According to Lompscher (1975), "flexibility of thought" expresses itself

> … by the capacity to change more or less easily from one aspect of viewing to another one or to embed one circumstance or component into different correlations, to understand the relativity of circumstances and statements. It allows to reverse relations, to more or less easily or quickly attune to new conditions of mental activity or to simultaneously mind several objects or aspects of a given activity. (p. 36)

The flexibility of thought is expressed by one' ability to:

1. reduce a problem to its essentials or to visualize it by using visual and structuring aids, such as informative figures, tables, solution graphs or equations (*reduction*),
2. reverse trains of thought or reproduce these in reverse, such as by working backward (*reversibility*),
3. simultaneously mind several aspects of a given problem or to easily recognize any dependencies and vary them in a targeted manner (e.g., by composing and decomposing objects, by working systematically) (*minding of aspects*),
4. change assumptions, criteria or aspects in order to find a solution, such as by working forward and backward simultaneously or by analyzing different cases (*change of aspects*), and
5. transfer an acquired procedure into another context or into a very different one by using analogies, for instance (*transferring*).

These manifestations of mental agility can be related to *heurisms*, which are known from the analyses of Pólya's approaches (1945/1973). *Heuristics* can be defined "as kinds of information, available to students in making decisions during problem solving, that are aids to the generation of a solution, plausible in nature rather than prescriptive, seldom providing infallible guidance, and variable in results" (Wilson, Hernandez, & Hadaway, 1993, p. 63). Moreover, not only the knowledge of different heuristics (flexibility of thought) is needed when problem-solving, but also self-regulatory abilities which evolve gradually through a 5-phase model (Zimmerman, 2002). It has been a long-term goal of mathematics educators to provide students with the skills necessary for success in problem-solving.

2.2 Reversibility

Problem-solving by working backward describes the ability to reverse trains of thought or reproduce these in reverse (e.g., Liljedahl et al., 2016; Pólya, 1945/1973). Other than when working forward, the target state forms the starting point in the solution process, whereas the calculated value forms the initial value of the problem. Thus, working backward leads to the entire thought process being reversed, since the task no longer corresponds to working forward (changing the direction of processing) (Bruder & Collet, 2011). Aßmus (2010a, 2010b) distinguished between two aspects of reversibility. On the one hand, the operation as such can be reversed (e.g., '+' becomes '–'), and, on the other hand, the sequence of the task processing can be reversed.

The ability to independently reverse trains of thought when working on mathematical problems has been recognized as one of the indicators for identifying mathematically gifted children (Käpnick, 1998). Consequently, the research on reversibility during problem solving has been primarily done with gifted and talented students. For instance, while working with potentially talented primary grade students, Aßmus (2010a, 2010b) investigated performance in heterogeneous primary mathematics lessons (Grade 2 and Grade 4) when solving reversing tasks. The results showed that the basic understanding of reversal was not present from the beginning. While many gifted Grade 2 students still had problems, on average, they were much more successful than the average children of the same age. For instance, no student from a control group was able to solve a problem with an unknown initial state, whereas at least 9% of the potentially gifted Grade 2 students (N = 182) succeeded in completing the tasks correctly, and in 35% of cases reasonable solution approaches could be identified. Symbolic tasks were generally processed backward more intuitively than word problems in which starting from an unknown initial state, various transformations needed to be performed in order to determine the initial state. Difficulty in the processing of the latter was maintaining the correct reversal of the operations or taking all operations into account when reversing.

On the other hand, Grade 4 students were more successful, with 36% of students reaching the correct solution (Aßmus, 2010b). Thus, reversibility was differently pronounced by primary grade students. Aßmus (2010b) hypothesized that this ability develops in the course of years with respect to average students (Aßmus, 2010b), though it may be more characteristic for gifted students (Aßmus, 2010a, 2010b).

The latter finding was supported by Amit and Portnov-Neeman (2016) who examined the effect of explicit teaching of problem-solving strategies, with a special focus being given to working backward strategy, on the ability of mathematically talented Grade 6 students to recognize and solve reversing tasks. The group that received explicit training showed higher results than the control group. Here, the students from the experimental group showed a better, clearer understanding of the strategy, and the strategic use improved over time (Amit & Portnov-Neeman, 2016). Moreover, they were more much resourceful in their solutions when solving a wide variety of reversing tasks. Amit and Portnov-Neeman (2016), therefore, confirmed the results of Aßmus (2010a, 2010b).

Whereas Aßmus (2010a, 2010b), and Amit and Portnov-Neeman (2016) focused on reversible thinking ability of gifted and talented students in the context of mathematics problem solving, Gullasch (1967) examined the relationship between mathematical problems and mathematical ability of Grade 7 students. His study revealed a high correlation between the reversibility of the mental activity, the level of school performance, and the ability to abstraction. Accordingly, the reversal of solution paths sets a basic level of mental activity.

2.3 Learning problem solving

In the field of problem-solving, there are two different approaches to learning heuristics. In an implicit heuristic training, it is assumed that the students internalize and unconsciously apply strategies they have learned through imitating practices of the teacher, and through sufficient practice. On the other hand, explicit heuristic training refers to making a given heuristic a learning goal, which is practiced step by step (e.g., Schoenfeld, 1985). Bruder and Collet (2011) pursued an explicit heuristic training focusing around Lompscher's (1975) idea of "flexibility of thought" in combination with self-regulation (Zimmerman, 2002), which consisted of the following five phases:

1. *Intuitive familiarization*: The teacher serves as a role model when introducing a problem to the students. This is achieved through moderation of behaviors by engaging in self-questioning (e.g., "What is the problem asking for?" "What information am I given?" "Am I headed in the right direction?") pertaining to different phases of the problem-solving process (before, during, and after) (Kuzle & Bruder, 2016). At this point, the heurism in focus is not specified.
2. *Explicit strategy acquisition:* The students get explicitly introduced to the heurism in focus by reflecting on the first phase, namely the particularities of the heurism get discussed, and the heurism is given a name. Here a prototypical problem for the heurism in focus is used so that the students can more easily recognize and use the heurism in future tasks.

3. *Productive practice phase:* The students practice the newly acquired heurism by solving different problems. These do not reproduce type problems, but rather expand the possibilities from the first two phases. Differentiation is a guiding concept so that students can choose at what cognitive level they want to work and adapt the observed learning behavior.k
4. *Context expansion*: The students practice the use of heurism in focus independent of a mathematical context. In that way, the students learn to flexibly, intuitively and independently of a context use the heurism in focus.
6. *Awareness of own problem-solving model:* The students reflect on their problem-solving process and document it.

Untrained problem solvers are often unable to consciously access the above-outlined flexibility qualities (Bruder & Collet, 2011; Liljedahl et al., 2016). In their research at the lower secondary level, Bruder and Collet (2011) were able to show that less flexible students (e.g., students with difficulties in reversing thought processes or transferring an acquired procedure into another context) profit from explicit heuristic training. Concretely, they were able to solve the problems just as well as more flexible students, who solved the problems intuitively. Thus, the problem-solving ability can be acquired through the promotion of manifestations of mental agility (reduction, reversibility, minding of aspects and change of aspects) in combination with self-regulation.

2.4 Design-based research in the context of SymPa project

Learning is a complex process, which depends on many factors, and thus, is difficult to control. Design-based research (DBR) as a research paradigm offers the opportunity to develop innovative teaching practices, and to develop context-sensitive learning environments. According to Wang and Hannafin (2005), design-based research is “a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researcher and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories” (p. 6-7). Hereby, they especially underline the flexible character of DBR and the importance of synergy of theory and practice, in contrast to other research paradigms.

The Design-Based Research Collective ([DBRC], 2003) lays down the cyclical and continuous nature of DBR comprising of design, enactment, analysis and re-design phase (see Figure 1). Under design is theory-driven development of a teaching-learning environment or a material understood, which will be implemented in the phase of enactment. In the next step, designed teaching-learning environment or material is analyzed in the evaluation phase. The improvements get then implemented in a re-design phase, and the cycle starts from the beginning on. In that manner, the result of any DBR approach is the development of new knowledge or suggestions on how to improve educational practice(s), such as exploring possibilities for creating novel learning and teaching environments, developing contextual theories of learning and instruction, advancing and consolidating design knowledge, and increasing the capacity for educational innovation (e.g., Collins et al., 2004; DBRC, 2003).

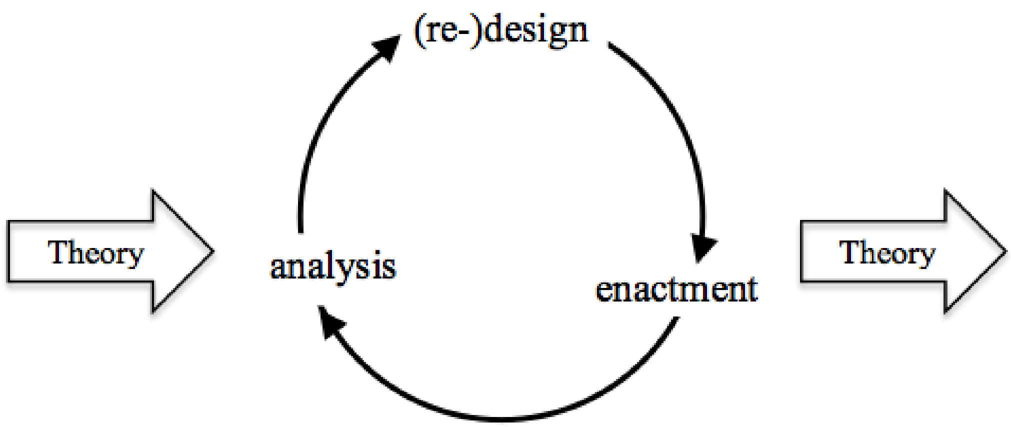


Figure 1. DBR cycle.

During the first seven DBR cycles of the SymPa project, the evaluation focused on analyzing to what extent are theory-based and practice-oriented problem-solving materials suitable and sustainable for their implementation in practice (Kuzle & Gebel, 2016) as well as on identifying the design elements contributing to improvement of the problem-solving competence (Kuzle, 2017a). Kuzle and Gebel (2016) reported that it was possible to develop a curriculum that met the local demands which enabled the implementation of problem-solving in practice. As a result, context-related design principles for the development of problem-solving material for Grade 6 students were developed (Kuzle, 2017a). The results showed that students needed an emotional incentive (hereby the figures) in order to be willing to solve problems and to prompt their reflective behaviors. Transparency of the material structure supported students' independent work, whereas material design (differentiation, transparent material structure with explicit reflections) was an important factor in the development of self-regulatory processes when problem-solving. Lastly, various design elements (text information, sample problem) allowed for explicit strategy acquisition and 3 to 4 problems seemed optimal for flexibility use. Moreover, Kuzle and Gebel (2016) demonstrated that DBR paradigm allowed creating novel teaching environments in which theory and practice were not detached from one another, but rather complemented each other (resource level of the educational design) (Collins et al., 2004). At the same time, the design was analyzed with respect to different objectives (e.g., language, level of performance, learning pedagogies) and subjective factors (e.g., school and personal influences) which inhibited full-implementation of the curriculum (institutional level of the educational design) (Collins et al., 2004). However, how these materials affect students' thinking and learning over time and subsequently their increase of knowledge in the context of mathematical problem solving remained open (cognitive level of the educational design) (Collins et al., 2004).

3 Research questions

On the basis of the above theoretical considerations and empirical results, the following research questions guided the study on problem-solving by working backward:

1. How do Grade 5 students solve reversing tasks before and after explicit heuristic training?
2. To what extent are Grade 5 students able to solve reversing tasks by working backward before and after explicit heuristic training?

4 Method

4.1 Research design and sample

For this study, an explorative qualitative research design was chosen. The study participants were Grade 5 students who showed interest in attending additional mathematics lessons on a voluntary basis that focused on problem-solving. In total 14 students (nine girls and five boys) from one rural school in the federal state of Brandenburg (Germany) participated in the study, and thus attended explicit heuristic training on the working backward strategy. Their performance in regular mathematics classes was good to very good.

4.2 Context of the study

The WoBa study (Problem Solving by Working Backward) was embedded in the SymPa project during which the students participated in explicit heuristic training which lasted about 7 months (see Figure 2). The explicit heuristic training took place once per week (45-minute lesson). During this period, the students received explicit problem-solving instruction pertaining to different heuristic auxiliary tools (informative figure, table), heuristic strategies (working systematically, working forward, working backward, analogy), and heuristic principles (composing and decomposing, invariance), which lasted two to three lessons per heurism. The lessons were taught by an experienced mathematics teacher.

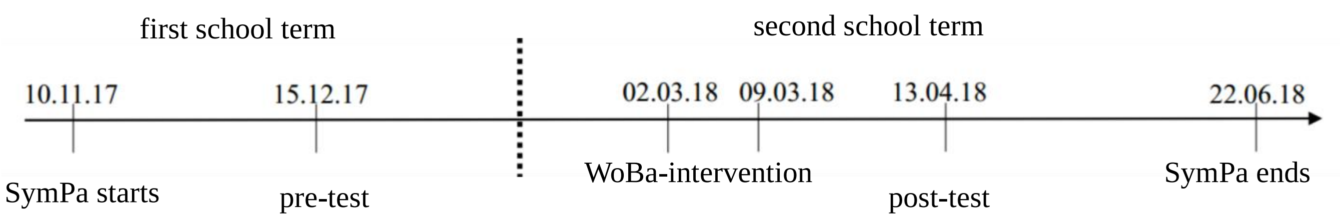


Figure 2. Timeline of the SymPa project.

During the explicit heuristic training, the students systematically learned heurisms using theory-based and practice-oriented materials (Kuzle, Gebel, & Conradi, 2017-2019) on the basis of the problem-solving teaching concept of Bruder and Collet (2011). The problem-solving material focusing on the working backward strategy is outlined below.

In the phase of intuitive familiarization, the students are given a representative problem for the working backward strategy (see Figure 3), which is solved together with the teacher, who serves as a moderator. Here the imitation of teachers' behavior takes place through self-questioning.

3.2 Working backward

3.2.1 Misplaced glasses

Oh Profi, where are your glasses?

I don't know. I must have misplaced them.
I've been thinking the entire time about what I've done today.

a) What does Profi mean by that?
b) How can he find his glasses again?

Figure 3. Introduction task on the working backward strategy (Kuzle et al., 2017-2019).

In the phase of explicit strategy acquisition, the working backward strategy gets formally introduced through a short student-centered information text (see Figure 4), and a partially worked out example (see Figure 5).

What is working backward?
Working backward is closely related to working forward, but runs in the other direction.
Here we start from the target state and follow the path to the initial state.
Questions, such as "What is wanted?", "What do I know about what I am looking for? "What do I need in order to find what I'm looking for?" offer orientation.

Figure 4. Information text on the working backward strategy (Kuzle et al., 2017-2019).

Example
Probi found this riddle in a magazine:

With a number between 1 and 9, six arithmetic tasks, starting with the upper result, are to be solved one after another in a clockwise direction in order to arrive at the final results of 136.

I'll try out some numbers.

That would take a really long time.
What strategy did we (just) learn here?
I would start with 136 and solve the task the other way round.

Figure 5. An example illustrates the working backward strategy (Kuzle et al., 2017-2019).

In what follows, at least three reversing tasks of different cognitive levels are presented (productive practice phase). This allows differentiation, where each student solves as many problems as he or she can. In addition, problems from different mathematical content areas are covered to allow transfer (context expansion phase).

Lastly, the tasks stipulate students to reflect on their problem-solving process (see Figure 6).

3.2.2 5-Aunts

Probi always gets candy from his aunts when he visits them. Each aunt gives him as much candy as he already has and one more. Probi has 5 aunts. After visiting all of them has 127 candies in his bag.

a) How many candies did he have before visiting them?
b) Have you been able to work backward on this task? Why did this strategy fit the task?
c) What heuristic tools did you use?

3.2.3 Number crusher

Input → Number crusher → Output

The "Number crusher machine" processes the numbers 1, 2, 3, and so on.
Even numbers are halved, uneven numbers are reduced by 1, e.g., 6 → 3 und 5 → 4.
The output number is then put back into the „Input" until 0 becomes the "Output", e.g., 5 → 4 → 2 → 1 → 0. So for 5 you would beed four steps (→) to reach 0.
Therefore, 5 is calles a 4-step number.

a) Examine how many steps you need for other numbers to reach 0!

For this you can use one of the learned heuristic tools.

b) How many 4-step numbers are there? List all of them.

Here by using a table, as a heuristic tool, can discover some interesting things.

3.2.4 Cutting paper

Profi and Probi play a game: Profi fold a piece of rectangular paper and then makes a straight cut. Probi only sees the end product and should find out how Profi folded and cut the paper.

a) Find out how Profi folded and cut the paper.
b) Did you work backward in part a)? Why?
c) What other heuristic strategies would fit you approach in a)?

3.2.5 Pouring water

Probi suggest Profi a bet and gives him two buckets:
„This is a 3-liter and a 5-liter bucket. They don't have any markings. You can now pour as much water back and forth as you want until you have exactly 4 liters of water in the 5-liter bucket. I bet you a hot chocolate."

a) Who will get the hot chocolate? Why?

Try to be systematic!
Think before you start solving the problem.

b) What heuristic tool did you use?
c) Try the informative figure and the table.
Which heuristic tool did you find best here? Why?

Figure 6. Further tasks with respect to the working backward strategy (Kuzle et al., 2017-2019).

From a design perspective, two figures, namely Profi (shape of an exclamation mark) and Probi (shape of a question), are introduced to support students' willingness to work hard. While Probi asks questions and gets stuck like every novice problem solver, Profi represents an expert problem solver who offers support to novice problem solvers (i.e., students).

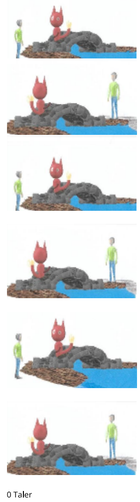
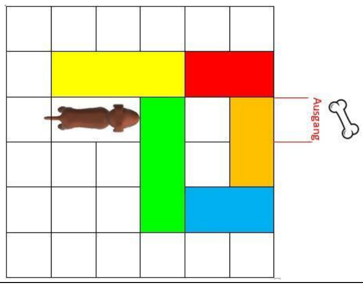
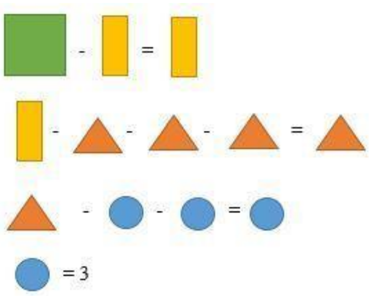
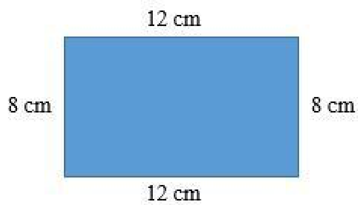
4.3 Data collection instruments and procedure

The study data consisted of (1) written data, (2) oral data, and (3) observations. (1) The written data were comprised of the students' solutions to pre- and post-test, each containing three problems addressing different mathematical levels. The pre-test was issued in December before that explicit training on working backward strategy at the beginning of the year started, whereas post-test one month after the end of the explicit training (see Figure 2). The students were allotted 45-minutes for each test. Both tests contained partially similar tasks in order to be able to compare the students' development of the working backward strategy use (see Table 1). The pre- and post-test included two types of reversing tasks. On the one side, the tests included tasks with the unknown initial state. In this case, starting from an unknown initial state, various transformations and the final state are described, and the initial state needed to be determined. The complexity of tasks varied based on the number of different operations, namely one ('Candy task') and two ('Four gates' task', 'Devil's task') operations. On the other hand, the tests included tasks that required a flexible reversal of relations. Thus, tasks which cannot be solved by working exclusively by working forward or backward, but whose processing requires flexible handling of relations which are often reversed several times, provided that they are not only tried out ('Circle task', 'Dogbone task', 'Rectangle task'). Additionally, the students reflected on different problem-solving strategies using reflection sheets at the end of explicit training.

(2) For the purpose of gaining a detailed insight into students' problem-solving processes, a brief interview (5-minutes) was conducted with four individual students, who were chosen on the basis of their results on the pre- and post-test. The following questions served as guidelines: "How did you come up with the solution?" "Do you think you could have solved solve the problem in another way?"

(3) During the explicit training on the working backward strategy, the researcher observed the lessons and made observation notes. Multiple data sources were used to assess the consistency of the results, and to increase the validity of the instruments.

Table 1. Pre- and post-test tasks.

Pre-test tasks	Post-test tasks
<u>Four gates' task</u> A man goes apple picking. To take his harvest to the town, he has to pass through four gates. At each gate, there is a guard and demands half of his apples and one more apple. In the end, the only thing left to the man is an apple. How many apples did he have at the beginning? 	<u>Devil's task</u> The devil says to a poor man: "Every time you cross this bridge, I will double your money. But every time you come back, you have to throw eight thalers in the water." When the man returned for the third time, he did not have a single thaler left. How many thalers did he have at the beginning? 
<u>Candy task</u> Marie gets a bag of sweets from her grandmother as a present. On the first day, she eats half of the sweets. On the second day, she eats half of the remaining sweets. Afterwards, she only had six sweets left. How many sweets were in the bag at the beginning?	<u>Dogbone task</u> The dog wants to get to his bone. Unfortunately, the way is blocked by colored blocks. Can you bring the dog to his bone? Find a way. 
<u>Circle task</u> What are the numbers for the remaining pieces? Solve the calculation.  	<u>Rectangle task</u> The sides of the blue rectangle are a total of 40 cm long. The blue rectangle is to become two rectangles. The sides of the two rectangles should be 40 cm long in total. a. What side lengths can the two rectangles have? b. Can you find any other solutions? 

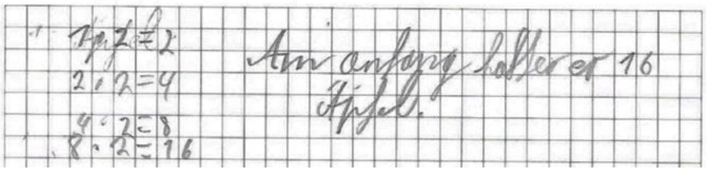
4.4 Data analysis

The analysis of the data was carried out in several steps. First, the data were examined with respect to used heurisms (1st research question). There was the possibility to solve the tasks with the help of working forward strategy (WF) or working backward strategy (WB). These could also have been processed without any specific strategy or not solved at all (NS). One coding was done per task, so that in the end, an overview was created, which reflected used problem-solving strategies of the respective student. The tasks were considered to be solved backward when the final result was recognized as the initial value, and the arithmetic operations of the task were correctly reversed ('Four gates task', 'Devil's task', 'Candy task', 'Circle task'). The 'Rectangle task' was also considered to have been solved backward, if it could be seen that the perimeter of the rectangle was divided by two and the result was distributed over the perimeter equation of a rectangle. The solution to the 'Dogbone task' was accepted as backward as long as it became clear that the block closest to the dog was moved. Subsequently, the inductive analysis of the problem-solving process was carried out by taking into account the different application performances of working backward strategy (2nd research question). In order to classify each student's achievement, these were assigned to individual levels of working backward (see Table 2). This was again carried out per task, in order to evaluate each student's progress in the project with respect to their ability to reverse trains of thought or reproduce these in reverse.

Table 2. A framework for different levels of working backward

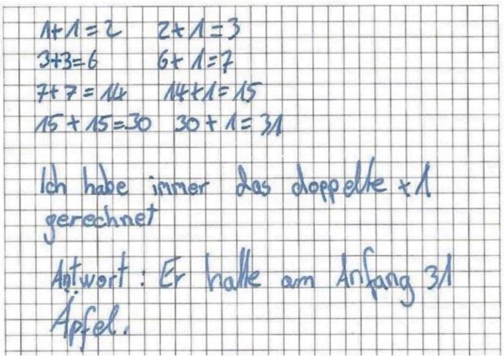
Levels of working backward	Description of students' behavior
WB1	Students do not use the given target state as a starting value for the calculation. The required operations are not reversed.
WB2	Students are able to use the given target state as a starting value. The required operations are not reversed.
WB3	Students can correctly reverse the required operations. However, the task is not calculated to the end so that the correct result is not achieved.
WB4	Students are able to work backward correctly.

Some students' solutions to the 'Four gates task' with assigned levels of working backward can be seen in Figures 7 to 9.



“At the beginning, he had 16 apples.”

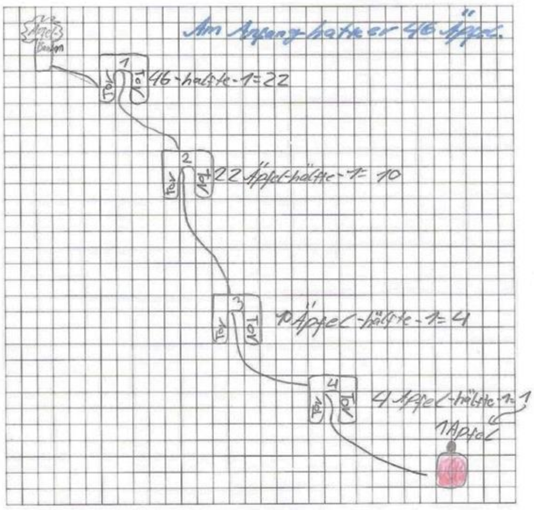
Figure 7. One student’s solution performing on WB1 level.



“I always calculated the double +1.”

Response: “He had 31 apples at the beginning.”

Figure 8. One student’s solution performing on WB2 level.



At the beginning he had 46 apples.
1. gate: “46 - half - 1 = 22”

2. gate: “22 apples - half - 1 = 10”

3. gate: “10 apples - half - 1 = 4”

4. gate: “4 apples - half - 1 = 1 apple”

Figure 9. One student’s solution performing on WB4 level.

The students’ self-reflection on different problem-solving strategies were also used to interpret the results as well as individual interviews. The latter was needed to correctly comprehend students’ problem-solving processes.

5 Results

5.1 Students' strategies when solving reversing tasks

In Table 3, it can be seen that the majority of the students intuitively used the working backward strategy on almost all pre-test tasks. One student only (#12) consistently used the working backward strategy on all three pre-test tasks. Additionally, two students (#5, #9) used the working forward strategy when solving the 'Candy task' and the 'Circle task'. Two students (#7, #14) did not employ any strategies when solving the 'Four gates task' and the 'Circle task'.

Table 3. Classification of the students' solutions on the pre-test in relation to used heurisms (• 'Four gates task', • 'Candy task', • 'Circle task')

	Student													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NS							•							••
WF					•				•			•••		
WB	•••	•••	•••	•••	••	•••	••	•••	••	•••	•••		•••	•

Similar results can be seen in Table 4 illustrating students' strategies on the post-test. Six students (#4-#9) employed the working forward strategy, whereas five of them when working on the 'Dogbone task'. Thereupon, it can be deduced that the reversal of thought processes has not proved to be a useful strategy for all students with respect to the 'Dogbone task'. Two students (#12, #13) did not employ any strategies when solving the 'Devil's task' and the 'Rectangle task'. Thus, after the explicit training, the majority of the students used the working backward strategy in most cases, and some were able to employ the most effective strategy for them.

Table 4. Classification of the students' solutions on the post-test in relation to used heurisms (• 'Devil's task', ○ 'Dogbone task', • 'Rectangle task')

	Student													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NS												•	•	
WF				○	•	○	○	○	○					
WB	•○•	•○•	•○•	••	○	••	••	••	••	•○•	•○•	○•	•○	•○•

Though Table 3 and Table 4 illustrate students' ability to (intuitively) reverse their thought processes when solving reversing tasks, there were differences on students' level of problem-solving by working backward which are reported on in the next section.

5.2 Students' level of working backward when solving reversing tasks

Based on different application performances of the working backward strategy on both pre- and post-test, the students' solutions were sorted into the appropriate levels of working backward. In that manner, a rough overview of the differentiated performance of each student was created (see Table 5). The performance of the students varied greatly, however, the majority (N = 11) were able to solve the 'Candy task' correctly (WB4), which could be solved by a simple reverse operation. The 'Four gates task' has shown to be more difficult for students. In total three students (#2, #4, #11) were able to solve the problem correctly (WB4) (see Figure 9), whereas eight students performed on WB1 or WB2 level (see Figure 7 and Figure 8). It may be that such poor performance was due to its complexity, as the task required two combined inverse operations. Nine out of fourteen participants were also able to work intuitively on the 'Circle task' performing on WB4 level, whereas two students were on WB2 level. Only one student (#14) was not able to solve the task. This symbolic task demanded not only the reversal of the existing operations but also an extension of the part-whole relation.

Table 5. Overview of students' solutions on the pre-test in relation to levels of working backward (• 'Four gates task', • 'Candy task', • 'Circle task')

	Student													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NS							• (grey)							• (grey) • (black)
WF					• (purple)				• (black)			• (grey) • (purple) • (black)		
WB1			• (grey)		• (grey)				• (grey)				• (grey)	
WB2	• (grey)	• (black)				• (grey) • (black)		• (grey)		• (grey)				
WB3			• (purple)											
WB4	• (purple) • (black)	• (grey) • (purple)	• (black)	• (grey) • (purple) • (black)	• (black)	• (purple)	• (purple) • (black)	• (purple) • (black)	• (purple)	• (purple) • (black)	• (grey) • (purple) • (black)		• (purple) • (black)	• (purple)

The students' performance on the post-test was more homogeneous than on the pre-test (see Table 6). Each student solved at least one problem correctly using working backward strategy (WB4). The 'Dogbone task' was solved by six and three students performing on WB3 and WB4 level, respectively. The task demanded a high degree of mental agility as it was an open task, which can be solved with both working forward and backward strategy. The 'Rectangle task' was successfully solved by nine students, whereas three students (#1, #4, #6) only showed rudimentary approaches to working backward (WB1). Compared to other tasks, it demanded geometric knowledge, namely calculating the perimeter of a rectangle and taking into account geometric ratio distribution. Despite the complexity of the 'Devil's task' due to the two combined inverse operations, eight students were able to solve the problem correctly (WB4), and two performing on WB3 level. Still, one student (#12) was not able to solve this problem, and three students showed were rudimentary approaches (WB1 or WF).

Table 6. Overview of students' solutions on the post-test in relation to levels of working backward (● 'Devil's task', ○ 'Dogbone task', • 'Rectangle task')

	Student													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NS												●	•	
WF				○	●	○	○	○	○					
WB1	●•			•		•				●				
WB2														
WB3			○						●	○	●○	○	○	○
WB4	○	●○•	●•	●	○	●	●•	●•	•	•	•	•	●	●•

Lastly, Table 7 illustrates more closely the students' performance on similar tasks, namely the 'Four gates task' and the 'Devil task'. At the beginning of the explicit training, the performance of eight students corresponded to the two lowest levels of working backward. These students only partially used the given target state as their initial value, and there was no reversal of the operations (see Figure 7 and Figure 8). After the explicit training, the results of the post-test showed that 10 students were able to solve the task partially correct (WB3) or correct (WB4). All of these students reversed both operations in a proper manner. However, two students forgot a subtask (one more crossing of the bridge) and for that reason did not achieve the correct end result. Only two students (#2, #4) were able to solve both tasks correctly (WB4) before and after the explicit training. Overall, it can be said that the ability to problem solve by working backward increased in seven of the remaining 12 students. For instance, students #7 and #14 did not initially show any strategic approaches. At the end of the explicit training, they were able to complete the corresponding task by using the working backward strategy. A direct comparison between the other tasks on both tests was not possible, because they were neither similar in context nor structure.

Table 7. Comparison of students' performance on similar tasks (grey • 'Four gates task', red • 'Devil's task')

	Student													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
NS							grey •					red •		grey •
WF					red •							grey •		
WB1	red •		grey •		grey •				grey •	red •			grey •	
WB2	grey •					grey •		grey •		grey •				
WB3									red •		red •			
WB4		grey • red •	red •	grey • red •		red •	red •	red •			grey •		red •	red •

The individual students' achievements are considered in Table 8 on the basis of a short interview. For this purpose, four children were selected, regardless of their gender. Additionally, Table 5 and Table 6 are compared and the self-assessment of the students is used as an interpretation aid.

Table 8. Assessment of the individual achievement when problem-solving by working backward

Student	Assessment of individual achievement with respect to the level of working backward
#2	In the pre-test, the student used the working backward strategy in all tasks, whereby she reached WB2 level once ('Circle task') and WB4 level twice ('Four gates task', 'Candy task'). In the post-test she solved all tasks by using the working backward strategy, performing on WB4 level. In the interview, it became clear that she already knew how to use the working backward strategy before the explicit training. For that reason, it was easy for her to solve the reversing tasks.
#8	In the pre-test, the students used the working backward strategy in all tasks, whereby he reached the WB2 level once ('Four gates task), and WB4 level twice ('Candy task', 'Circle task'). The results of the post-test showed performance improvement when solving similar tasks (from WB2 to WB4 level). In the interview, he reported that he found it difficult to work backward and was often confused when working on reversing tasks.
#12	In the pre-test, the students used the working forward strategy in all tasks. On the other hand, two tasks in the post-test were solved by using the working backward strategy, namely 'Dogbone task' (WB3) and 'Rectangle task'(WB4). Her solution to the 'Devil's task' did not allow any conclusions with respect to used problem-solving strategy. In the interview, she reported that she often got confused when working on the reversing tasks, and it was difficult for her to reproduce her thoughts in reverse.
#14	In the pre-test, the student worked on two tasks ('Four gates task', 'Circle task') without using a specific heuristic strategy, whereas she solved the 'Candy task' by using working backward strategy (WB4). In the post-test, she solved two tasks correctly by using the working backward strategy, namely 'Devil's task' and 'Rectangle task' (WB4). She also reversed her thoughts when working on the 'Dogbone task', but reached WB3 level only.

Reflection sheets of all students on the topic of "heuristic strategies" provided additional information on the extent to which they found the working backward strategy useful. Four students rated the strategy as easy, whereas seven students as difficult. The rest abstained from reflecting on the strategy. Moreover, six students reported that they would use this strategy in the future. Self-reflection of four selected students was consistent with their performance on the pre- and post-test.

6 Discussion and conclusions

For almost one school year, the students took part in explicit heuristic training. Within the WoBa study, a special focus was given to problem-solving by working backward, and the students' ability to work backward was evaluated. At the beginning of the explicit training, almost all students' approaches showed instances of working backward. They intuitively reversed their thought processes when solving the reversing tasks. Overall, every student used this strategy in the post-test, with as many tasks as in the pre-test were not solved backward. There may be various reasons for this. The students used both working forward and backward strategy when solving the logic task ('Dogbone task'). Consequently, no arithmetic or geometric skills were required, but only the reversibility of the trains of thought. On the other hand, this task can also have been perceived as difficult, since such a prototypical task was not dealt with within the explicit training (see Figures 3-6). Should the former be the case, then this would be a distinguishing feature with regard to mental agility and with it developed mathematical ability (Gullasch, 1967). Simple reversing task ('Candy task'), as well as the symbolic task ('Circle task'), were intuitively solved by using the working backward strategy compared to complex reversing tasks ('Four gates task', 'Devil's task').

Before the explicit training, the students were at different stages of development with respect to reversibility. Although the students found it difficult to reverse trains of thought and reproduce operations in reverse, the results show improvement with respect to different levels of working backward during problem-solving. After the explicit training, students were able to further develop their skills and reach the next stage of development (Bruder & Collet, 2011; Wygotski, 1964). It can be also assumed that the students were able to detach the structure of the heuristic strategy from the task context and transfer it to similar tasks (Lompscher, 1975). This became particularly evident when comparing students' performance on similar tasks, namely the 'Four gates task' and the 'Devil's task'. Consequently, it can be concluded that mathematically-interested students could train their mental agility, and thus, compensate for deficits in the area of mental activity by using the heurisms imparted by the SymPa project (Bruder, 2014; Kuzle & Bruder, 2016; Lompscher, 1975). Moreover, the ability to reverse thought processes or to reproduce these in reverse when confronted with reversing tasks may not only be reserved for gifted students (Aßmus, 2010a, 2010b) but also mathematically-interested students can successfully develop this ability. Also, the willingness to apply the heuristic strategies, which were mentioned in the students' self-reflection, helped promote the intellectual ability and compensate for deficits (Bruder, 2014; Lompscher, 1975).

In the SymPa project we have consciously decided to develop theory-based materials which were evaluated in a school setting in order to simultaneously (1) (develop) and evaluate the suitability of practice-oriented materials, (2) to develop a sustainable problem-solving teaching concept, and (3) to gain insights into individual students' learning processes. The latter relates to the cognitive level (Collins et al., 2004) of DBR. The results have shown that a synergy of the teaching concept and the problem-solving material allowed mathematically-interested students access to problem-solving, specifically to working backward strategy.

Despite positive results in the context of problem-solving by working backward, some drawbacks need to be discussed. This study was an exploratory qualitative study using a specific sample in the context of additional mathematics lessons on problem-solving on a voluntary basis. Hence, the results are limited to mathematically-interested Grade 5 students. Additionally, a small sample was used, so not all processes were reported. These limitations suggest a possible next step in research. Since the problem-solving materials were developed for Grade 5 and 6 students, future studies may look into the extent to which they are implementable in regular mathematics lessons rather than in special mathematics contexts. Since the tasks cover different mathematical areas, they may be implemented flexibly. Moreover, the effect of the materials on the development of all students' problem-solving competence, not only with respect to the strategy of working backward, is an area highly important to investigate taking into consideration mathematics standards worldwide (e.g., FNBE, 2004, 2014; KMK, 2005; NCTM, 2000). It is also questionable whether a long-term intervention on the subject of working backward would have influenced the students' results or promoted their mental agility in the area of logic tasks. Additionally, the pre- and post-test tasks were selected according to the mathematics curriculum (RLP, 2015). During the post-test, however, it became clear that not all students were proficient in calculating the perimeter of rectangles, as this was not the content of the previous school year. This aspect had a significant influence on the results of the post-test. Lastly, the pre- and post-test were deliberately structured in such a way that they did not only consist of similar tasks in order to avoid routine processing of the tasks. However, a deeper insight into the development of reversibility between pre-/post-testing would have been provided by other similar tasks. This direction may be fruitful for future studies.

The SymPa project demonstrated the fruitfulness of the synergy between practice (i.e., school, practitioners) and theory (i.e., research, university staff). From the perspective of practice, the school gained high-quality material on problem-solving which allowed supporting needs of students interested in mathematics. From the perspective of theory, a framework for different levels of working backward was developed. This may also be used by practitioners in order to evaluate students' levels of working backward as well as to promote their development of reversibility of thought. Additionally, the study findings reflect a great potential for problem-solving in school mathematics. Both the developed materials using design-based research-approach and the teaching of heurisms in the classroom stipulated the development of students' flexibility of thought when problem-solving by working backward. The majority of students improved their ability to work backward, progressing to the next or second next level. In addition, almost all students reached the highest level of working backward. Thus, the study results show that the theory-based and practice-oriented materials using DBR approach not only allow sustainable implementation in practice (Kuzle, 2017a, 2017b) but also promote the development of targeted problem-solving abilities. Further research, however, is needed to evaluate the utility of the materials with respect to general problem-solving ability.

That DBR as a research paradigm may support gradual improvements in both practice and theory, and that with it further theoretical and practical developments are possible, preclude no doubts.

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