How did you solve it? – Teachers' approaches to guiding mathematics problem solving

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This case study focuses on teachers' actions during problem-solving lessons. The aim of this study was to find out how teachers guide students during mathematics problem-solving lessons: What kinds of questions do teachers ask? How do students arrive at solutions to problems? The dataset contained videotaped fourth-grade math lessons in which students solved a mathematical problem. The research reveals that teachers can guide students in numerous ways and possibly in ways that prevent students from searching for their own solution strategies. For this reason, problem-solving exercises alone are not sufficient for teaching problem solving for students, teachers must also be instructed in how to properly guide students. In the conclusion section, we discuss the types of questions that enable teachers to promote active learning in students, which should be the goal of instruction according to the constructive learning theory.

Keywords: problem solving, guiding problem-solving, teachers' questions, active learning, activating guidance

1 Introduction

In a fast-changing world, it is difficult to determine what kind of knowledge or skills students will need in the future. Even so, students will likely need to solve complex problems with possibly more than one solution. One approach in problem solving involves helping students develop the type of thinking needed for solving problems rather than simply instructing students on how to solve problems (Hakkarainen, Lonka, & Lipponen, 2008, 14). Problem solving skills do not improve only by listening; the student must be activated. This is called active learning. (Bonwell & Eison, 1991)

Problem solving is a process in which current knowledge is applied in a novel way (cf. Kantowski, 1980). Problem-solving tasks usually involve non-standard problems in which the solver does not instantly know the solution or the correct solving strategy (Pehkonen, 2004). The most common problem-solving strategies are systematic listing, simplification of the problem, finding a pattern, trial and error, deduction, generalisation of the problem, solving the problem backwards, and progressing

through a familiar problem (Schoenfeld, 1985; Pólya, 1945; Leppäaho, 2007; LeBlanc, 1977).

Problem-solving tasks alone do not inform the problem-solving process. In school, teachers substantially affect students' problem-solving processes (Pehkonen, 1991, pp. 24–25), and teachers can guide students in many ways (Stigler & Hiebert, 2004). One central goal of teachers should be to develop students' persistence in solving problems. The atmosphere where teachers support students in investigating and finding new solutions or solving strategies can influence students' ability to solve problems (Näveri, Ahtee, Laine, Pehkonen, & Hannula, 2012, pp. 81–82). Developing students' problem-solving skills is one central goal in the Finnish national curriculum for comprehensive schooling and is one central issue in international educational science (National Board of Education [NBE], 2016; Organisation for Economic Co-operation and Development [OECD], 2013; Binkley, Erstad, Herman, Raizen, Ripley, Miller-Ricci, & Rumble, 2012).

In this study, we researched which type of teacher guidance would increase students' ability to solve problems, and also investigated the effects of teacher guidance on students' solution strategies. Our aim was to explore the kind of teacher questioning that promotes active learning in students. The importance and the novelty of this paper is that we make a summary of probing, guiding and factual questions that promote active learning.

2 Theoretical framework

In this section, we discuss problem solving and teachers' role in problem solving based on previous studies.

2.1 Guiding problem solving

Teachers can greatly influence students' progression in mathematical problem solving. Asking the right questions or leading students to think about a particular problem on a higher level can help students' problem-solving experience become more productive (Pehkonen, 1991; Stigler & Hiebert, 2004; Hähkiöniemi & Leppäaho, 2012).

Usually, in a classroom, problem-solving lessons have three different phases: 1) First, the teacher introduces the task by showing the problem and motivating students

to work on it. Also, the teacher makes sure that everyone has understood the instructions. 2) Then, students try to solve the task while the teacher helps and supports them. 3) Finally, at the end of the lesson, the 'looking-back' phase reviews students' achievements in solving the task (e.g. Lampert, 2001; Stein, Engle, Smith, & Hughes, 2008; Hähkiöniemi & Leppäaho, 2012; Laine, Näveri, Hannula, Ahtee, & Pehkonen, 2011).

While students are solving tasks, teachers should support their mathematical self-confidence by providing positive but also realistic feedback (Linnenbrink & Pintrich, 2003). A teacher can help students by listening to them carefully and by flexibly taking into account their needs. Empathic listening, when practiced by a teacher, helps students to think aloud their own ideas and enables teachers to support students without giving too much advice (Pehkonen & Ahtee, 2006). Sometimes, if a teacher reveals too much information about the problem at hand, a nonstandard problem may turn into a standard task (Tzur, 2008; Swan, 2007).

Ultimately, teachers can guide students in a variety of ways (Pehkonen, 1991; Stigler & Hiebert, 2004). Son and Crespo (2009) studied how 34 prospective teachers analysed hypothetical student solutions in addition to how these teachers responded to students. They found two different types of teacher guidance: *teacher-focused* and *student-focused.* Teacher-focused guidance occurred when teachers considered the hypothetical solutions provided by students, for example, when a teacher commented on what was wrong with students' solutions or how students could justify or improve their solution. Meanwhile, in student-focused guidance, teachers provided students with opportunities to investigate and to justify the solutions on their own.

Hähkiöniemi and Leppäaho (2012) also studied different levels of teacher guidance as evidenced by prospective teachers' actions during problem-solving tasks. They found three levels of teacher guidance:

1. Teachers who practiced *surface-level guidance* did not consider meaningful aspects of students' proposed solutions and instead gave advice or comments that were unrelated to students' solutions.
2. Teacher who provided *inactivating guidance* noticed relevant aspects of students' proposed solutions and guided students towards these aspects while simultaneously revealing the right answer.

3. Finally, teachers who employed *activating guidance* noticed relevant aspects of students' proposed solutions, connecting these to the problem at hand while also encouraging students to further explore these aspects.

In this context, the aspects most relevant to problem solving deal with the adequate use of problem-solving methods given the task at hand: these may involve justifying the solution, examining other solutions, generalising or building connections, for example.

Son and Crespo (2009) in addition to Hähkiöniemi and Leppäaho (2012) examined prospective teachers in hypothetical problem-solving situations. In this study, we extended their research by categorising data collected from real elementary classrooms. By doing this we receive information about how their classification works in a real situation.

2.2 Teachers' questions

Teachers often guide students by asking them questions (Harrop & Swinson, 2003). Several types of questions can be asked: some question may ask about facts, while others lead students to think about problems on a higher level (Sahin & Kulm, 2008; Myhill & Dunkin, 2005; Harrop & Swinson, 2003). A large number of studies have researched the questions used by teachers during problem solving (e.g. Sahin & Kulm, 2008; Myhill & Dunkin, 2005; Harrop & Swinson, 2003; Harri, Hähkiöniemi, & Viiri, 2012; Laine, Näveri, Kankaanpää, Ahtee, & Pehkonen, 2014; Martino & Maher, 1999).

Although these studies have categorised teachers' questions in different ways, several main types of questions have also been identified by numerous authors. For example, a teacher may focus on fact-based questions by asking, for example, 'What is five plus five?' Usually, this type of question is close-ended with only one right answer. Another type of questioning is that which leads students to think about problems on a higher level. For example, teachers can ask students to justify the solution. Finally, other teacher questions might focus on helping students to progress, such as 'What could you do next?' Finally, some categories of questioning appear to be out of context, such as those related to the organisation or the management of a lesson.

Table 1 shows how different authors have categorised teacher questions and their relation to one another. The rows denote the categories used by each study, and the columns further unite these the different questions into similar types. The middle

columns represent the types of questions that can lead students to thinking about the problem on a higher level and that can help students progress when solving problems.

Table 1. Different categories of teacher questioning during problem solving. The rows indicate the categories identified by different studies, and the columns unite these categories under similar concepts.

	Fact-based questions		**Questions that promote higher-level thinking**	**Questions that help students progress**	**Other questions**
Sahin & Kulm (2008)	factual questions		probing questions	guiding questions	
Harri et al. (2012)	factual questions		probing questions	guiding questions	other questions
Myhill & Dunkin (2005)	factual questions		speculative questions	process questions	procedural questions
Harrop & Swinson (2003)	of fact	closed solutions	open solution	task supervision	routine

Factual or closed-solution questions refer to facts. Myhill and Dunkin (2005) defined these questions as those that invite a predetermined answer. Harrop and Swinson (2003) separated these questions into two categories: 1) *factual* (of fact), or questions that inquire about academic information, and 2) *closed-solution questions*, or questions related to the problem-solving context that only have one right answer.

Guiding, process or *task supervision questions* are related to the problem-solving process. *Guiding questions* help students to progress (Sahin & Kulm, 2008). *Process questions* invite students to explain their aloud thinking or learning process (Myhill & Dunkin, 2005). Finally, *task supervision questions* verify that the task is being solved, for example, 'How will you measure that?' (Harrop & Swinson, 2003).

Probing, speculative or *open-solution questions* lead students to think on a higher level, for example, 'What could that mean?' or 'What do you think might happen then?' There is more than one right answer to these questions (Sahin & Kulm, 2008; Myhill & Dunkin, 2005; Harrop & Swinson, 2003; Harri, Hähkiöniemi, & Viiri, 2012).

Procedural, routine or *other questions* relate to the organisation and the management of the lesson and not specifically to the aims of the lesson. For instance, teachers can ask 'Where are you going?' or 'Can you all see?' (Myhill & Dunkin, 2005; Harrop & Swinson, 2003; Harri, Hähkiöniemi, & Viiri, 2012).

In particular, probing questions are an essential aspect of guiding problem solving. Martino and Maher (1999) studied these questions in greater depth. They found six types of probing questions, including those that 1) estimate students' understanding, 2) direct students' attention to a vague or incomplete part of their argument, 3) cultivate students' interest in the problem, 4) encourage mathematical justification, 5) direct students to examine other students' solutions and 6) encourage generalisation of a solution based on similar problems. These types of questions were also considered in Sahin and Kulm's (2008) category of probing questions, although they placed questions that direct students' attention to vague or incomplete aspects of their argument into the category of guiding questions.

3 Research questions

In this paper, our aim was to discover how teachers guided students during problem-solving lessons in which students were instructed to solve a non-standard problem. We wanted to find out what kinds of questions teachers asked and how students arrived at solutions. Finally, we were also interested in understanding how teachers' guidance and students' solutions were related to one another. Our research questions are as follows:

1. How do teachers guide students during problem-solving lessons?
   (a) What level of guidance do teachers provide?
   (b) What kinds of questions do teachers ask?
2. What kinds of solutions do students produce?

4 Methods

The study is part of the broader Finland-Chile research project financed by the Academy of Finland. This experiment on teacher-guided problem solving focused on an experimental group of teachers and students in the Helsinki metropolitan area. Teachers gave problem-solving lessons to their students once a month on average between the years 2010 and 2013. All lessons were videotaped, and students' solutions were collected. In this study, we focused specifically on how teachers guide students with questions.

The analysis is based on qualitative research methods. We analysed the videotaped problem-solving lessons and the students' solutions to the given tasks by using

deductive content analysis (Seale, Gobo, Gubrium & Silverman, 2004). In this study, we specifically researched lessons in which fourth-grade students had to solve a digit-time task. In this non-standard task, the aim was to find times on a 12-hour clock for which the sum of the four digits was six (e.g. 03:03).

All the teachers in this study, Paula, Tina and Mia (pseudonyms), are female and had worked as teachers for several years. Before the problem-solving lesson, the teachers met and discussed their understanding of the given problem task. The teachers could still decide how they wanted to organise the lesson and how they presented the task to the students. They did not receive any advice from the researchers with respect to the central aspects of the problem or how to guide students.

During the 45-minute lessons, one of the researchers (LN) recorded the teachers' work. The videos were then transcribed, and the teachers' questions and guidance were categorised (AK). Finally, two of the researchers (AL & LN) performed a parallel coding of the material.

The teachers' guidance were categorised into the three categories developed by Hähkiöniemi and Leppäaho (2012). The categories are *activating guidance*, in which a teacher guides students to investigate relevant aspects of the problem without revealing the right answer; *inactivating guidance*, in which a teacher notices relevant aspects of students' solutions but at the same time reveals the right answer; and *surface-level guidance*, whereby the teacher does not notice relevant aspects of students' solutions but instead provides comments that are unrelated to students' solutions.

Table 2 presents examples of how teachers' guidance was categorised.

Table 2. Examples of different levels of teacher guidance.

Example	Interpretation	Category
"How can you be sure that you have found all the different number series and their different variations?" (Paula)	The teacher guided the student to investigate the relevant aspects of the task without revealing the right answer.	Activating guidance
"Did you notice that we have to operate with a 12-hour clock? Now, this time is on a 24-hour clock." (Tina)	The teacher noticed the relevance of the student's comment but also revealed why the solution is incorrect.	Inactivating guidance
The student: *"Can this be something like sixty?"* (in reference to minutes)	The teacher does not notice the relevant aspects of the student's solution.	Surface-level guidance
Mia: *"No, I guess. There is maybe too much something there. But you have to work together with Lisa."*	Instead, she gives comments that are unrelated to the student's solution.	

Teachers' questions were categorised into four categories developed by Sahin and Kulm (2008) and Harri, Sironen, Hähkiöniemi and Viiri (2012). *Probing questions* lead students to think about the problem on a higher level. *Guiding questions* help students to proceed with problem solving. *Factual questions* ask about facts yet do not help students proceed. Finally, *other questions* were outside the problem-solving context. Sometimes, teachers asked more than one factual question when those questions helped students to proceed. In this case, the sequence of factual questions was defined as belonging to the category of *guiding questions*. Table 3 present how the teachers' questions were categorised.

Table 3. Sample of teachers' questions

Example	Interpretation	Category
Have you developed a strategy to find these times? (Paula)	This question led the student to think about the problem on a higher level.	probing question
There, you have 00:06, very good. Are there any other options that begin with 00…? (Tina)	This teacher guided the student to investigate relevant aspects without revealing the right answer.	guiding question
Who knows what a 12-hour clock is? (Tina)	This teacher asked about a fact that does not help the student progress towards the solution.	factual question
Jonas, would it be easier to concentrate if you did not sit so close to Oskar? (Paula)	This question was out of context and was related to the organisation of the lesson.	other question

We also categorised students' answers according to the applied solution strategies and the obtained results. We found two types of strategies: systematic listing and trial and error. The students did not use systematic listing all the time. In this case, there were some examples where students found new times that fit the task criteria by simply changing the position of the numbers. In the solutions using trial and error, there was no clear aim or systematic listing. Some solutions also revealed instances where students had tried wrong answers and then erased them.

5 Results

Next, we will present the results. We will start by describing how teachers' lessons were structured and how teachers guided students. Then, we will show what kinds of questions teachers asked and what kinds of solutions students obtained.

5.1 Teachers' guidance

All lessons were 45 minutes long and contained three phases: introduction to the task, solving the task, and reflecting on the task. Depending on the teacher, the emphasis placed on these different phases varied.

Case 1: Paula – Awakener of thinking and active listener

At the beginning of the lesson, Paula briefly introduced the task and carefully explained the concept of a 12-hour clock. Paula also motivated students by telling them that the student with the most correct answers would get a surprise.

Paula's students worked alone while she walked around the classroom and asked questions. She placed emphasis on finding a strategy to solve the task. She asked every student what kind of strategy she or he had used.

> Paula: Oh, you have found already [the solution] many times. What kind of strategy did you use?
> Student: I changed the positions of the numbers.

Paula also made the students think about invalid solutions that used times outside the 12-hour clock. She usually promoted active learning during this inquiry because she did not reveal the right answer.

> Paula: What time of the day do you have here?
> (The student rubs out the answer.)

> Paula: Why is it not valid?
> Student: Because it is afternoon.

Also, the students posed questions to Paula. Usually, Paula answered by posing a counter-question, which led the students to reason the problem by themselves. This is also an example of guidance that encourages active learning.

> Student: Can I organise the same numbers in a different way?
> Paula: Is it the same time, then?
> Student: Yes, I mean, if there is one-four-five and five-four-one, is it okay?
> Paula: If the sum of the numbers is six. Is it?
> Student: No.

Paula's guidance mainly promoted active learning because she directed students' attention to invalid solutions and guided students towards investigating the assignment without revealing answers. Paula also encouraged students to find a solving strategy. In Paula's lesson, the phase of 'solving the task' was emphasised.

Case 2: Tina – Encourager and motivator

Tina started the lesson by showing a chest (a small box) and telling students that the student with the most correct answers could open it. She then briefly introduced the task.

In the task-solving phase, the students worked in small groups while Tina encouraged and motivated the students to find new times. In fact, she was so excited that she sometimes proposed new times to the students.

> Tina: Could you use that 00: and something?
> Student: Hmm...
> Tina: Maybe not anymore. Well, how about if you start with 01? Do you have all those times already? 01:05 – do you have that?

She also once accepted a wrong solution only because she wanted the student to feel successful.

> Tina: Try to find two numbers which sum up to six.
> Student: Can I write o-o-six-o?
> Tina: Well, why not. Actually, it is then o-one-o-o, but you can put that. It is still a very good solution.

Despite her motivation and encouragement, Tina usually guided students in a way that did not promote active learning – she largely noticed the relevant aspects of

students' solutions but also often revealed the right answer. In Tina's lesson, positive motivation and joy upon reaching the findings were emphasised.

Case 3: Mia – Idea provider and solution inspector

Mia started the lesson by introducing the task. She took about 20 minutes, much longer than the other teachers. In the introduction, Mia told students what kind of strategy they should use, preventing students from searching for their own solutions. Mia guided the students to list all the different decompositions of the number six and then marked the solutions that corresponded to correct times. This strategy was complicated and difficult.

While students solved the task in small groups, Mia walked around in the classroom and guided the students to use the strategy that she had explained.

> Student: Should we put 0: something . . .
> Mia: Yes, or then you could do like I did earlier – take that six and think about how we can decompose it into four numbers.

She also checked and corrected the students' solutions and suggested new times to the students.

> Mia: I will give you a new one – try to use four-one-one. These numbers, because you can split six like four-one-one. Could you find some good times by using these numbers?

Mia mostly guided the students in way that discouraged active learning because she usually told students what they were supposed to do and then provided them with justifications. Also, surface-level guidance was often provided. Sometimes, she did not notice the relevant aspects of students' solutions but talked about things unrelated to the task.

> Student: Can this be something like sixty . . . ? (talking about minutes)
> Mia: No, I guess. There is maybe too much something over there. But you have to work together with Lisa.

Also, more than once Mia had corrected the students' solutions wrong on the solution sheets, which can also be considered surface-level guiding.

5.2 Teachers' questions

The number and the quality of the teachers' questions varied a lot. Paula asked questions about 50% of the time that she was speaking, while Tina asked questions 33% of the time and Mia only 15% of the time. Table 4 summarises the teachers' questions.

Table 4. Summary of the questions asked by teachers

	Probing questions	Guiding questions	Factual questions	Other questions	Sum
Paula	22 (41%)	18 (33%)	11 (20%)	3 (6%)	54 (100%)
Tina	6 (11%)	13 (24%)	31 (57%)	4 (7%)	54 (100%)
Mia	4 (12%)	6 (19%)	13 (41%)	9 (28%)	32 (100%)

As one can see, Paula mainly asked probing and guiding questions. Tina mainly asked factual questions, while Mia asked factual and other questions. These results are in line with the level of activating guidance provided by the teachers.

Probing questions usually checked the level of students' thinking, for example, 'Have you developed a strategy to find these times?' (Paula). Some questions also guided students to justify their own solutions or to investigate other students' solutions, such as 'Did someone else utilise the summation?' (Tina).

There were also several types of **guiding questions**. Paula mostly asked guiding questions related to solving strategies or invalid solutions, for example, 'Try to find a strategy like Paul did. Could it be easier then?' Unlike Paula, Mia and Tina asked guiding questions whose aim was only to find the next right solution, such as 'Have you used, for example, the numbers four-one-two-o?' (Mia).

Factual questions were usually related to the number of solutions, like 'How many have you found?' (Paula). Also, other factual questions inquired about single times (Tina: 'Do you have one-o-o-five?'), students' work (Mia: 'Have you done this by turns?') or specific conceptions (Tina: 'What does a 12-hour clock mean?').

Mia asked also many questions that were unrelated to the problem-solving context. For example, she asked 'Do you want to stay here?' and 'Are you discussing?' These questions were usually meant to refocus students on the task.

5.3 Students' solutions

We found two types of students' solving strategies: systematic listing and trial and error. All of Tina's students and almost all of Paula's students used the systematic solving strategy. Roughly half of Mia's students did not use the systematic solving strategy. The first picture represents a solution using systematic listing and the second picture a solution where the student has tried numerous options.
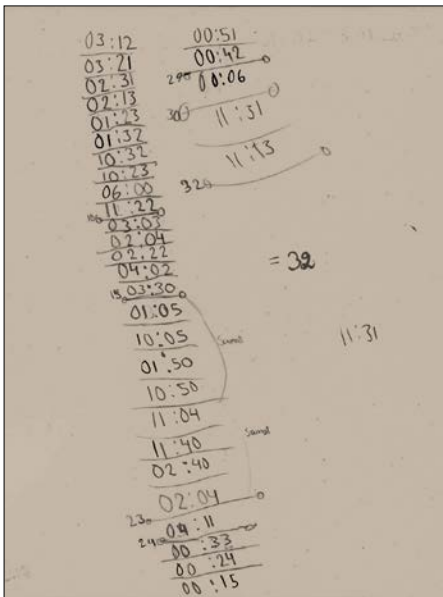

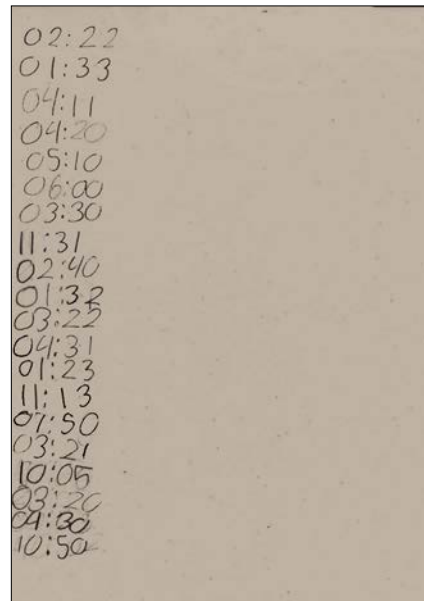Figure 1. A solution sheet using systematic listing with 31 correct solutions.

Figure 2. A solution sheet using trial and error with 17 correct solutions.

In total, 38 correct solutions exist for the given task. None of the students was able to find all of the correct times. Tina's and Mia's students worked in pairs. Paula's students worked alone. The average number of correct solutions was highest among Tina's students, the average being 26. Paula's students had almost equally good results, 24 right solutions on average, whereas the corresponding number for Mia's students was 20.

Variation in the number of right solutions was highest among Mia's students (12–35). The number of right solutions varied a little less among Paula's and Tina's students' (the ranges being 15–31 and 23–35, respectively).

5.4 Summary of the results

We studied three different teaching styles employed by teachers to guide problem solving. We also studied how students arrived at solutions and whether these were correct. Table 5 summarises the levels of guidance provided by teachers, the type of questions asked by teachers and the number of correct solutions provided by students.

Table 5. Summary of the main results per teacher

	Paula	**Tina**	**Mia**
Guidance level	Mainly activating	Mainly inactivating	Mainly inactivating and surface-level
Teachers' most-used questions (2 categories)	probing questions: 22 (41%) guiding questions: 18 (33%)	factual questions: 31 (57%) guiding questions: 13 (24%)	factual questions: 13 (41%) other questions: 9 (28%)
Mean number of correct student solutions	24	26	20
Range of correct student solutions	15–31	23–35	12–35

The solutions of Mia's students varied a lot: one group had 35 right answers, but more than half of the groups had less than 20. Although Mia revealed the solving strategy to the students, the results were not as good as those of the other two teachers. Mia's solving strategy was complicated, so the students with the best scores did not actually use it.

Paula and Tina asked many more questions than Mia. Their students also got good results. However, Tina asked a lot of questions, which helped her students to find new times, and she sometimes even revealed right answers to the students. Paula helped and encouraged the students in an active way, enabling them to find a solving strategy. She did not only emphasise finding the correct times.

One clear difference among the students' solutions per teacher was the number of wrong solutions. Mia's and Tina's students listed many wrong solutions, but Paula's students had only a few wrong solutions. This supports Paula's frequent direction of students' attention to invalid solutions. Also, unlike Mia and Tina, Paula carefully explained the concept of the 12-hour clock, which played a central role in this task.

According to the students' solutions (Table 5), the use of a systematic solving strategy seems to affect the number of correct solutions. Sometimes, a more systematic solving strategy was evident after students had listed several solutions by trial and error. Thus, a sufficient time frame is necessary for students to solve problems and to encounter the correct strategy. Sometimes, the solving process has to be restarted from the beginning, which takes a lot of time.

6 Discussion

Students' positive results are not always good indicators of students' learning. Therefore, it is important to consider the learning process from different perspectives (cf. Tzur, 2008; Swan, 2007). This study is in an agreement with previous studies that an introductory phase prior to problem solving is important in order to carefully explain to students the relevant concepts surrounding a task (cf. Näveri et al., 2012). Moreover, teachers must be careful not to reveal the solving strategy to students or help them too much because this can turn a nonstandard problem into a standard task (cf. Stigler & Hiebert, 2004; Tzur, 2008; Swan, 2007). Thus, the best method could be that the teacher carefully introduce the task and then guide students depending on their level.

According to this study, there is a connection between the number of teacher questions and the level of guidance provided by teachers (cf. Sahin & Kulm, 2008; Hähkiöniemi & Leppäaho, 2012). The teacher who asked many probing and guiding questions also guided students in a way that promoted active learning, while the teacher who asked fewer questions did not guide students in an active way.

However, the number of questions asked does not necessarily equate activating guidance; the quality of the questions also matters. Based on this study, we made a summary of questions that can help teachers properly guide students in order to promote active learning (Table 6). This broadens the understanding of teachers' questions. All the guiding questions do not activate the students' thinking and some factual questions can promote active learning.

Table 6. Teacher questions for promoting active learning (All the examples are from this study.)

	Example	Notes
Probing questions	'How did you solve this?' 'What is the strategy you have used?' 'How did you end up with this solution?'	- Usually lead students to explain their own thinking or ideas. - Indicate that the teacher is interested in students' ideas.
Guiding questions	'What could you solve next?' 'What is said in the assignment?' 'What does this mean? Why do you think it is not valid?'	- Should direct students' attention to invalid solutions but also ask students to justify their thinking. - Can include counter-questions that lead students to think about the problem in new ways or to justify the solution – these are a valid response when students ask questions related to the task.
Factual questions	'How far have you progressed?' 'How many solutions have you found?'	- Can motivate students because they indicate that the teacher is interested in the students' work. - Alone do not promote active learning because they do not encourage students' independent thinking to progress.

Activating probing questions usually lead students to explain their own thinking or ideas. Generally, students must answer these questions with more than one word. These questions also indicate that the teacher is interested in the students' ideas, for example, 'How did you solve this?'

Activating guiding questions encourage students to progress without revealing the answer. If a teacher guides students' attention to invalid solutions, then the teacher should also ask why the solution is not valid. Moreover, if a student inquires a teacher about something, the teacher can ask a counter-question that leads the student to think about the problem in a new way or to justify the solution.

Activating factual questions motivate students because they indicate that a teacher is interested in students' work. However, these questions alone do not correspond with activating guidance because they do not encourage students to progress in their thinking process.

Because the amount of research data considered in this study is not very large, it is not possible to make generalisations. It is also important to remember that in real classroom situations many things in addition to teachers' guiding and questions affect

students' actions. That is why students might have problems in the task despite of the teacher's guiding.

In the future, it would be interesting to study how the results of teacher guidance might change if teachers are instructed to consciously ask more questions that would promote active learning. We would also like to understand how such a change would influence students' work, particularly their solutions and attitudes towards mathematics.

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