LUMAT Special Issue 2022: Mathematical Thinking and Understanding in Learning of Mathematics

Student teachers' common content knowledge for solving routine fraction tasks

Anne Tossavainen

Department of Health, Learning and Technology, Luleå University of Technology, Sweden

This study focuses on the knowledge base that Swedish elementary student teachers demonstrate in their solutions for six routine fraction tasks. The paper investigates the student teachers' common content knowledge of fractions and discusses the implications of the findings. Fraction knowledge that student teachers bring to teacher education has been rarely investigated in the Swedish context. Thus, this study broadens the international view in the field and gives an opportunity to see some worldwide similarities as well as national challenges in student teachers' fraction knowledge. The findings in this study reveal uncertainty and wide differences between the student teachers when solving fraction tasks that they were already familiar with; two of the 59 participants solved correctly all tasks, whereas some of them gave only one or not any correct answer. Moreover, the data indicate general limitations in the participants' basic knowledge in mathematics. For example, many of them make errors in using mathematical symbol writing and different representation forms, and they do not recognize unreasonable answers and incorrect statements. Some participants also seemed to guess at an algorithm to use when they did not remember or understand the correct solution method.

Keywords: common content knowledge, elementary school, fractions, student teacher, teacher education

ARTICLE DETAILS

LUMAT Special Issue
Vol 10 No 2 (2022), 256–280

Pages: 25
References: 36

Correspondence:
anne.tossavainen@ltu.se

https://doi.org/10.31129/LUMAT.10.2.1656

1 Introduction

Teaching and learning of fractions has shown to be a challenging area in mathematics (e.g., Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Löwing, 2016; Ma, 2010; Newton, 2008). As Lamon (2007, p. 629) expresses, fractions like ratios and proportions are "the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science." Nevertheless, fractions are an essential part of school mathematics and an important part in the development of algebra and proportional reasoning. Elementary school students' knowledge of fractions and division can even predict their algebraic skills and performance in mathematics several years later (Siegler et al., 2012).

A deep understanding of rational numbers requires knowledge of different fraction interpretations such as the operator model and linear models (see e.g., Kieren, 1993; Lamon, 2007, 2020). However, student teachers seem to favor the part-whole model that has traditionally been connected to fractions and taught in elementary schools, and they struggle with other fraction interpretations (Lamon, 2020; Olanoff et al., 2014). Developing skills with fractions also requires the ability to perform fraction operations and to build up some degree of fraction sense. According to Lamon,

> This means that students should develop an intuition that helps them make appropriate connection, determine size, order, and equivalence, and judge whether answers are or are not reasonable. Such fluid and flexible thinking is just as important for teachers who need to distinguish appropriate student strategies from those based on faulty reasoning. (Lamon, 2020, p. 143)

In Sweden, the national curriculum for the compulsory school states the core content related to fractions first as parts of a whole and as parts of whole numbers, which should be compared and named as simple fractions in grades 1-3 (Skolverket, 2011). Further, in grades 4-6, the knowledge requirements include an understanding of rational numbers in fraction, decimal and percentage form. The main calculation methods for fractions are included in the curriculum for grades 7-9. Even though efforts have been made to improve learning results in mathematics, studies show that Swedish elementary school students still have deficiencies in fulfilling the above knowledge requirements (Löwing, 2016; Skolverket, 2016, 2019). Therefore, it is also important to focus on student teachers and to study their knowledge of fractions thoroughly.

Previous studies (e.g., Ma, 2010; Tirosh et al., 1998; Zhou et al., 2006) have shown the important role of teacher education in developing student teachers' fraction knowledge and the need for further research and international comparisons in this topic (Olanoff et al., 2014). The present study is a part of a more comprehensive research project that seeks to respond the research needs in this field by expanding the view to the Swedish teacher education context. The aim of this paper is to investigate Swedish elementary student teachers' common content knowledge (CCK) of fractions by analyzing errors and difficulties in their solutions for routine fraction tasks. The research question of this study is:

How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks?

2 Previous research on student teachers' fraction knowledge

A number of studies investigating different aspects of student teachers' fraction knowledge have been published in mathematics education research. Olanoff et al. (2014) present a summary of 43 research articles focusing on student teachers' mathematical content knowledge in the area of fractions. These studies conducted, e.g., in Australia, Taiwan, Turkey and in the USA between the years 1989 and 2013, show that student teachers' fraction knowledge is relatively strong in performing fraction procedures. However, when including all basic operations of arithmetic and using basic fraction tasks that can be found in elementary school mathematics textbooks some studies also show limitations in student teachers' knowledge of fraction operations (e.g., Newton, 2008; Young & Zientek, 2011).

For example, Newton (2008) identified several error patterns when studying elementary student teachers' knowledge of routine fraction tasks in the USA. For addition, and especially when the denominators were different, the most common error was adding across numerators and denominators. In the subtraction of fractions, student teachers had difficulties changing forms, they subtracted across and left blank. In multiplication, they made whole-number errors with mixed numbers, cross-multiplied fractions instead of multiplying across, kept the common denominator in the answer, added numerators or denominators, and made errors in changing forms as well. Student teachers in Newton's study were most uncertain about dividing fractions, and even more error patterns were found for that operation: (a) finding a common denominator and keeping it in the product, (b) leaving blank, (c) reciprocals, (d) flipping the dividend instead of the divisor, (e) making mistakes with whole number facts, (f) cross-dividing or cancelling, and (g) adding or subtracting numerators or denominators. Newton (2008) concluded that the most common error in the operations with the routine fraction tasks was keeping the denominator the same even though it was not suitable.

A few years later, Young and Zientek (2011) showed that student teachers' competence vary by fraction operation; division and multiplication are the most difficult operations for student teachers. Moreover, student teachers' knowledge of fraction operations was partly rule-based and, for example, they tended to overgeneralize the rule of converting fractions to have like denominators for multiplication as well. Many of the student teachers' error patterns seemed to be based on incorrect memories of algorithms they had learned before which led them to inappropriate use of procedures; in some tasks they used correct procedures and in some other tasks with the same operation they chose the incorrect ones. Thus, Young and Zientek (2011) concluded that student teachers in their study were not able to accurately judge their abilities to correctly perform the fraction operations.

Previous research have also reported on student teachers' difficulties understanding the meanings behind fraction procedures and why the procedures work (e.g., Ma, 2010; Marchionda, 2006; Olanoff et al., 2014). Tirosh (2000) concludes that many student teachers in Israel are not capable of explaining the fraction division procedure even though they are able to use it. Similarly, the American final-year student teacher in Borko et al.'s study (1992) showed a weak understanding of both multiplication and division of fractions at the end of her teaching practice after completed a mathematics methods course; her knowledge of fraction division was based on a rote understanding of the invert-and-multiply algorithm and she lacked any knowledge of other representations such as visual representations of fractions she could use to demonstrate the division solution. Moreover, student teachers seem to lack flexibility in moving away from procedures and using fraction number sense, for example, when converting a fraction to a decimal (Muir & Livy, 2012; Olanoff et al., 2014). This may be one reason many student teachers have difficulties solving fraction story problems and creating their own fraction word problems (e.g., Ball, 1990; Tirosh, 2000; Toluk-Uçar, 2009).

Researchers in previous studies have also concluded that the relationship between student teachers' conceptual and procedural knowledge of fraction operations is weak, and that their fraction knowledge reflects the misconceptions that children have when working with fractions (e.g., Lin et al., 2013; Van Steenbrugge et al., 2014; Young & Zientek, 2011). Similar to children, many student teachers make errors based on prior knowledge of whole numbers, and when misapplying algorithms, especially the multiplication algorithm, student teachers' errors can also relate to their prior knowledge of fractions, e.g., to cross-multiplying which can be used when comparing fractions (Newton, 2008).

Student teachers are assumed to have a certain level of competence in using fractions when they are admitted to teacher education. However, Van Steenbrugge et al. (2014) concluded that one reason Flemish student teachers perform at a low level with fractions is the limited time spent on fractions in teacher education. Teacher education does not seem to have an impact on student teachers' common content knowledge of fractions, which reveals a need to develop mathematics teaching in this area (Van Steenbrugge et al., 2014).

Even though the multiple challenges related to the teaching and learning of fractions are widely recognized in many international studies as shown in the examples above, there seem to be few recent studies focusing on student teachers' fraction knowledge in the Nordic countries. One such study focuses on Icelandic student teachers' mathematical content knowledge showing that they have considerable difficulty with fractions; their knowledge is procedural and relates to "standard algorithms" learned in elementary school (Jóhannsdóttir & Gíslandóttir, 2014). A study of Norwegian student teachers (Jakobsen et al., 2014) shows that they have difficulties when solving fraction word problems; the student teachers seem to lack familiarity with mathematical notions of fractions, and they have difficulties interpreting elementary students' solutions and giving sense to fraction solutions different from their own. Furthermore, a study conducted in Finland indicates that a large number of those applying for teacher education have challenges in solving fraction algorithms (Häkkinen et al., 2011). As stated in many previous studies in the field, researchers in these Nordic studies as well highlight teacher educators' responsibility in ensuring the quality of student teachers' fraction knowledge and the need for further research in this area. The present study contributes to the field by taking the topic to Swedish teacher education and presenting an analysis of student teachers' CCK of fractions in a Swedish context. This gives an opportunity to see some worldwide similarities and national challenges in student teachers' fraction knowledge.

3 Theoretical framework

Over the last few decades, an increasing research interest has been given to subject matter knowledge as an important part of teaching (e.g., Shulman, 1986). In his original work, Shulman (1986) suggests three categories of teacher knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Subject matter knowledge includes not only the knowledge of the content of a subject area but also knowledge of substantive and syntactic structures. By these, Shulman refers to the varying ways the basic concepts, principles and facts of a discipline are organized and identifies the legitimate rules in that domain. Successful teaching requires also pedagogical content knowledge, what Shulman (1986) calls "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

In mathematics, there has been a lack of agreement about definitions, language, and basic concepts within teaching-specific mathematical knowledge (Hoover et al., 2016). Ernest states that

> the teacher's knowledge of mathematics is a complex conceptual structure which is characterized by a number of factors, including its extent and depth; its structure and unifying concepts; knowledge of procedures and strategies; links with other subjects; knowledge about mathematics as a whole and its history. (Ernest, 1989, p. 16)

Many studies concerning student teachers' knowledge base have focused on the differences between their conceptual and procedural knowledge (e.g., Lin et al., 2013; Marchionda, 2006). Conceptual knowledge is knowledge that is rich in relations (Hiebert & Lefevre, 1986). When it comes to fractions, it includes the understanding of the definition of fractions and other relevant number sets, fundamental facts about these numbers, and how the essential facts are related in the context of fraction tasks. Procedural knowledge about fractions concerns computational skills that are needed for solving fraction tasks and familiarity with the proper ways to denote fractions and their operations, for example, how to use appropriate rules and notations for the division of fractions (Hiebert & Lefevre, 1986). Maciejewski and Star (2016) conclude that flexible procedural knowledge is a key skill, which can be a way to improve students' conceptual knowledge as well. However, as Newton (2008) states, "dichotomizing mathematical knowledge into procedures and concepts does not account for its complexity" (p. 1105).

Even though teacher knowledge base has been regarded as an essential part of effective teaching, scholars have argued whether and how it contributes to students' learning. Thus, several studies have been conducted to examine the extent to which, for example, the mathematical knowledge for teaching framework (MKT) relates to learning (e.g., Charalambous et al., 2020). When analyzing the mathematical demands of teaching, Ball et al. (2008) identified the mathematical knowledge that is needed for teachers to effectively perform their work. They present the MKT framework based on Shulman's (1986) knowledge categories by using domains of subject matter knowledge and pedagogical content knowledge, and suggest that many teaching tasks included in the subject matter knowledge domain require mathematical knowledge that is not dependent on the content in the pedagogical domain.

This study focuses on the common content knowledge (CCK) category of the subject matter knowledge domain. Ball et al. (2008) define CCK "as the mathematical knowledge known in common with others who know and use mathematics" (p. 403). This knowledge and skill are used in a wide variety of settings in day-to-day work, and is thus not unique to teaching. CCK can be regarded as a basic competence in mathematics since it includes, e.g., performing calculations correctly, carrying out mathematical procedures, recognizing wrong answers, and using definitions, terms and notations correctly as well as understanding fractions (Ball et al., 2008). CCK covers mathematical tasks and questions that can be answered by anyone with a general knowledge of mathematics.

A robust CCK is a requirement for specialized content knowledge (SCK), which contains mathematical knowledge and skills that are used in teaching settings and are typically not needed for purposes other than teaching. As Ma (2010) concludes, "in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it" (p. 71). SCK includes abilities like explaining why common denominators are used when adding fractions and what is the procedure behind the invert-and-multiply algorithm in dividing fractions or determining whether a nonstandard approach would work in general to solve a given problem (Ball et al., 2008). In other words, this is knowledge of how to make mathematics understandable to students. However, in some cases it can be difficult to differ CCK from SCK. For example, detailed knowledge of different fraction representations such as symbolic and pictorial representations can be regarded as specialized knowledge, but it can also be common knowledge for others in their daily work (Ball et al., 2008).

Based on Ball et al.'s (2008) description of CCK, the present study investigates elementary student teachers' CCK by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. The concept error is chosen for this study instead of, e.g., misconception or misunderstanding, and its definition for this study is presented later in this paper. The concept difficulty is also used since it was assumed that not all findings in the analyzed fraction solutions could be categorized as obvious errors. However, this paper does not intend to explain why specific errors appear. As Radatz (1979) states, "errors in the learning of mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is such a close interaction among causes" (p. 164).

4 Methodology

4.1 Participants

The participants in this study were 59 university students in Swedish elementary school teacher programs, which are meant for to prepare teachers for the preschool class and grades 1-6 of compulsory school. Most of the participants were in the third academic year of their four-year programs, and they had already passed their first mathematics course in teacher education. One of the key aims of this mandatory course is to deepen student teachers' mathematical knowledge and strengthen their computational skills. During the first mathematics course, fraction content that is studied before entering teacher education and included in the curriculum for the compulsory school, e.g., calculating with fractions by using all operations, simplifying, reducing and extending fractions, and converting fractions to decimal, percent and mixed number forms, is recalled and repeated with all student teachers. At the time of the present study, the participating student teachers were starting their second mathematics course, which had a focus on the didactics of mathematics.

4.2 Data collection

Data for this study were collected by using a printed questionnaire. The voluntary participants were given 90 minutes to answer it before the first lecture of their mathematics didactics course at the university campus. They were asked for some background information (part 4 in the questionnaire), to write about the concept of fraction (part 1), and to describe how they might teach a fraction addition task to elementary school students (part 2). This paper focuses on six routine fraction tasks that were included in the questionnaire as well (part 3, see Appendix A). The instruction for the tasks was presented as follows: 'Calculating with fractions. Solve the following tasks as well as you can without using a calculator. Show all the steps you use.' With the instruction 'show all the steps', the participants were indirectly guided to show their fraction knowledge using mathematical algorithms, which they had been repeating in the previous mathematics course in teacher education and which can be regarded as CCK for mathematics teachers. It was also possible to use other representations such as pictures or decimal forms since the instruction was written: 'Solve the following tasks as well as you can'. Detailed knowledge of fractions and their correspondence to different representations is also knowledge that mathematics teachers need in their daily work (Ball et al., 2008).

The six fraction tasks used in this study were based on similar tasks that can be found in Swedish mathematics books and support materials for grades 4-6 mathematics. All four operations, i.e. addition, subtraction, multiplication and division, were included in the tasks with different types of fraction content: (a) addition with common denominators, (b) addition with different denominators, (c) subtraction with different denominators, (d) subtraction with a whole number, (e) multiplication with different denominators, and (f) division by a whole number. The participating student teachers were already familiar with this kind of tasks, and the tasks were defined as routine tasks since the operations were written without any context (c.f. Newton, 2008).

4.3 Data analysis

In this study, elements from Radatz's (1979) information-processing classification were used to categorize the errors in the participants' solutions. Three error types were of interest in the analyzed routine tasks: errors that are due to (1) lacking knowledge of prerequisite skills, facts, and concepts, (2) incorrect associations or inflexibility in thinking, and (3) application of irrelevant rules or strategies. Radatz (1979) states that category (1) "includes all deficits in the content- and problem-specific knowledge necessary for the successful performance of a mathematical task" (pp. 165-166), and he continues by elaborating "Deficits in basic prerequisites include ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of necessary concepts and symbols" (p. 166). The error type (2) includes negative transfer from similar tasks even though the conditions for the tasks are different. In the last category, the errors are mainly based on successful experiences when applying comparable rules or strategies in other content areas. However, making a clear distinction between those error types mentioned above is often difficult because many of the causes interact during the learning process (Radatz, 1979).

When analyzing the participants' solutions, the answers were first coded as correct or incorrect. As a correct answer, it was assumed in this study that the answer was converted to a mixed number when possible or that it was presented in the simplest fraction form. This decision was based on the instructions and examination of the previous mathematics course, which the participants had passed in their teacher education. In Sweden, simplifying and extending fractions are considered to be prerequisite skills for the addition and subtraction of fractions (Löwing, 2016). Thus, giving the answers for fraction tasks in the simplest fraction form or as a mixed number is also encouraged in Swedish compulsory school mathematics books, and it is often expected that the answers are primary given as a fraction and not as a decimal or percent, which might be mathematically correct as well. However, providing an answer in these forms was not mentioned in the instruction since in another part of the questionnaire it was examined whether the participants were able to provide these fraction-related concepts themselves.

After the first round of coding, a qualitative analysis focusing on the solution methods was conducted. It was investigated whether there were solution methods used other than mathematical symbol representations and what kind of errors were included in the solutions. However, using other methods than mathematical algorithms was not classified as an error. Following Young and Zientek (2011), errors were defined as technical and procedural errors, where the latter consist of obvious errors in using fraction operations. This was in the cases where the participants were misusing the procedures, for example, adding across numerators and denominators in addition. This refers to Radatz's (1979) first error type. Also, if their methods seemed inefficient or misleading when used in teaching settings, and if there seemed to be a lack of number sense or negative transfer from similar tasks in the solutions, the operations were classified as including errors in this study. For example, this was done in the cases where the participants were using unnecessary long solution methods or big common denominators, or they were using common denominators when unnecessary. This classification has a connection to Radatz's error type (2) presented above.

Before deciding on the final error categories, the errors were coded several times to ensure the reliability of the coding. The rating of the errors was also discussed with an additional researcher and after that, the primary errors were coded by using symbols E1, E2, E3 etc. (see Appendix B). The errors were categorized altogether as seven error types. Three of them are related to fraction operations (procedural errors): errors in addition or in subtraction (E5), errors in multiplication (E6), and errors in division (E7). These categories include several subtypes of errors that were made by individual or multiple students.

Technical errors in this study are related to presenting the answer (E1), mathematical writing (E2), mathematical facts (E3), and leaving the task blank in the research questionnaire (E4). These errors include also solutions that can be regarded as correct in contexts other than this study. For example, E1 category consists of five subcategories that describe the solutions, which were counted to be incorrect in the context of this study even though the answers may otherwise be mathematically correct, i.e. presenting the answer as decimal or not as a simplified fraction form. E2 includes partial computation and missing solution steps as well as illogical mathematical symbol writing, and E3 consists of minor errors in calculation. Despite mathematical writing errors in the procedures, the participants' solutions have been counted as correct in the analysis if they produced a correct answer for the fraction task.

The findings of the study were analyzed in terms of the number or percentage of participants who successfully performed the fraction tasks by providing correct answers or those who made errors in their solutions. Otherwise, the main data analysis was based on a qualitative description of the student teachers' solutions for the tasks. When analyzing the solutions, the anonymous participants were given number codes according to the order their questionnaires were analyzed. These number codes are used as references in the figures presented in the next section.

5 Results

The research question of the study 'How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks' will be answered next. This section begins by describing the participants' fraction solutions in general; their errors and difficulties with the different tasks will then be described in more detail.

5.1 On student teachers' solutions for the routine fraction tasks

Table 1 shows the number of student teachers giving correct answers for the fraction tasks, using pictorial representations and making the most common technical errors E1, E2 and E4. As can be seen in Table 1, there is a wide difference between the student teachers when solving the routine fraction tasks. Two of the 59 participants gave correct answers to all six tasks, whereas on the other end of the spectrum, there were participants that gave only one or not any correct answer. The participants with all correct answers used mathematical symbol representations and wrote their solution steps in the algorithms in such a way that it was easy to follow the procedures they used. The participants with the least correct answers made errors with all operations, and they had difficulties in simplifying the fractions and converting them to mixed numbers. Only one of these student teachers seemed to demonstrate knowledge in using the different algorithms and writing the mathematical steps; otherwise, the participants with the least correct answers did not seem to notice the errors they made with the operations.

Table 1. A summary of the participants' solutions for the fraction tasks

Number of correct answers	Number of participants	Number of participants using pictures	Number of participants making errors in presenting the answer (E1)	Number of participants making mathematical writing errors (E2)	Number of participants leaving blank (E4)
6 (all correct)	2	0	0	0	0
5	11	2	4	8	2
4	17	3	8	13	3
3	16	2	14	10	3
2	5	0	3	3	4
1	4	0	4	3	1
0	4	0	2	1	3
Total	**59**	**7**	**35**	**38**	**16**

In general, the participating student teachers did not show a robust CCK in presenting mathematical algorithms and solutions steps. Almost a half of the participants failed to follow the instruction to show all their solution steps at least with one of the tasks. This may indicate that they had difficulties in mathematical symbol writing or that they did not notice where or how to write more details in their solutions. This was most common in the case of division where only six participants presented a logical mathematical solution by using fractions. For example, the step showing how to do the change to common denominators is missing in the next solution even though the mathematical writing is done correctly and the right answer is found: $\frac{4}{5}+\frac{2}{3}=\frac{12}{15}+\frac{10}{15}=\frac{22}{15}=1\,\frac{7}{15}$. Furthermore, the participants using pictorial representations did not present any steps with their solutions. However, they provided more often the correct answers for the tasks than those who used mathematical algorithms incorrectly in their solutions.

Moreover, the participants' CCK in using different representations in their fraction solutions seemed limited. Some participants used decimals but they made errors in giving correct answers; one of them used decimals for all the tasks without ending to any correct answer. Pictorial representations were used most often to solve the division task. The multiplication task $\frac{3}{4} \cdot \frac{2}{5}$ was not solved with pictures, which may indicate that the multiplication procedure is more challenging to present with pictures than the other fraction operations in the analyzed tasks. Also, it seemed that the participants used pictures for the tasks that were easier to visualize with pie charts; for example, not for the addition task with different denominators 5 and 3. Moreover, when the participants used two separate circles for subtraction, the circles (pie charts) in their solutions seemed to represent the fractions rather than the subtraction procedure (see Figure 1). In the case of addition, two circles can easier be used to illustrate the procedure as well (see Figure 2). To summarize, it seemed that the participating student teachers' CCK knowledge for using pictorial fraction representations to demonstrate solution procedures was limited.

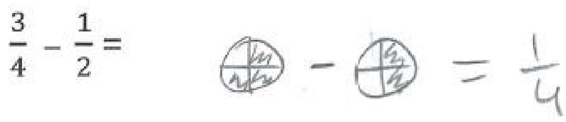

Figure 1. A subtraction solution with pie charts (participant 16)

Figure 2. An addition procedure illustrated with pie charts (participant 22)

Many participants also made different kinds of obvious E3 errors in their solutions. These errors in mathematical facts did not seem to be directly related to fractions but were rather simple mistakes in calculation, like 12+10=24 and 3·3=6. Some participants also made multiple error types in their solutions, e.g., they used illogical mathematical writing for a wrong solution method and made calculation mistakes as well (see Figure 3).

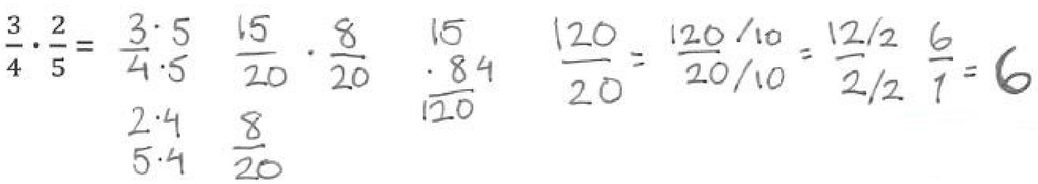

Figure 3. A solution with multiple errors (participant 41)

The most common technical error types were E1, E2 and E4 (see Table 1). Most of the participants who had difficulties in mathematical writing (E2) did not use mathematical notations correctly throughout their solutions; many of them used the equal sign incorrectly presenting their solutions often as separate calculations and ignoring whether the equal sign was written between the solution steps or not. The thinking model behind these solutions can often be understood, but mathematically, this kind of partial writing results in illogical statements (see Figures 3 and 4).

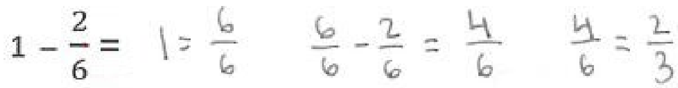

Figure 4. A solution with illogical writing (participant 27)

Several participants also used the division sign incorrectly and, in particular, they seemed to have difficulties in making a distinction between dividing and simplifying the fractions with their notations (see Figure 5).

Figure 5. Examples of errors in using the division sign (participants 47 and 48)

It seems that the participants who provided the solutions above were using division while meaning to simplify the fraction $\frac{6}{20}$, which should have led to an answer that was different from the one they provided. However, some participants were able to use the mathematical notations correctly, writing, for example: $\frac{3}{4} \cdot \frac{2}{5} = \frac{6/2}{20/2} = \frac{3}{10}$.

More than half of the participating student teachers made errors concerning the proper form for the answer (E1), and their uncertainty and illogical use of different fraction forms could be found in many solutions: in some tasks they provided the answer as a simplified fraction or a mixed number while in other similar cases, they did not. If neglecting these technical E1 errors, the total number for correct answers in the tasks would have been greater; still, it would not have led to all answers correct in any of these routine fraction tasks, and only seven participants would have correctly solved all tasks.

Several participants also left at least one of the tasks blank. This leaving blank error (E4) was made in all types of the fraction tasks except addition with common denominators, and it was most common for multiplication and division, which were both left blank by 10 students. Leaving blank may indicate uncertainty in using the procedures when the participants did not remember the correct algorithms.

5.2 Student teachers' errors and difficulties with the routine fraction tasks

The number of participants giving correct answers and making different error types E1-E7 in the analyzed six fraction tasks are summarized in Appendix B. An analysis of their errors and difficulties with the fraction tasks is presented below in the same order as the tasks existed in the questionnaire.

Addition with common denominators: $\frac{2}{3}+\frac{2}{3}$. Altogether, 42 participants (71%) gave the correct mixed number answer for this task. Five of them showed detailed steps in their solutions, writing, for example, $\frac{2}{3}+\frac{2}{3}=\frac{2+2}{3}=\frac{4}{3}=1\frac{1}{3}$. Some participants may have perceived this task so simple that there was no need to show detailed solution steps, and four participants used a pictorial representation (circles or rectangles) as a method to find the correct answer.

Most errors here were technical E1 errors. Six participants gave the answer as an improper fraction $\frac{4}{3}$ instead of converting it to a mixed number, and one participant gave the answer as a decimal, i.e. 1.33. Four student teachers seemed uncertain and wrote their mixed number answers within parentheses or as an unfinished answer in two parts $1+\frac{1}{3}$ or they gave even two alternative answers, $\frac{4}{3}$ or $1+\frac{1}{3}$. Moreover, nine participants made a procedural E5 error by adding across the numerators and denominators. After adding incorrectly, three of them also simplified the fraction $\frac{4}{6}$ to $\frac{2}{3}$ without noticing that this was not a reasonable answer when adding $\frac{2}{3}+\frac{2}{3}$.

Addition with different denominators: $\frac{4}{5}+\frac{2}{3}$. Compared with the first addition task, a smaller amount of participants, 37 of 59 (62%), performed correctly this task. Those who had difficulties in the previous task made similar E1 errors in presenting the answer here as well. One participant converted his/her improper fraction solution again to a decimal number (1.466). However, all these participants as well as those with the answer in the correct mixed number form, showed their mathematical solution steps: they found the common denominator for the given fractions and used a proper solution method. Some participants made minor computational errors (E3), and two student teachers left this task blank (E4). Interestingly, the other of them solved correctly the previous task (addition with same denominators) and the next one (subtraction with different denominators). Thus, it seemed that he/she was uncertain about the role and use of denominators in these fraction tasks.

Technical error E2 was the most common error type here since several participants used incorrect mathematical notations, and had partial computations or missing solution steps. However, there were even more procedural E5 subtype errors in the addition operation. Ten participants used a total of seven different faulty methods for the addition operation, which led to as many different incorrect answers. Three of these student teachers found the common denominator 15, but they multiplied only the denominators, adding the fractions as follows: $\frac{4}{15}+\frac{2}{15}=\frac{6}{15}$. One participant used an unnecessarily large common denominator, 30, instead of 15. Even though mathematically correct, this method seemed inefficient and it can also be interpreted as a lack of number sense. Two participants added across numerators and denominators; the other of them did this even though he/she did not add the like denominators in the first addition task. Four participants used varying multiplicative methods, for example, they cross-multiplied or multiplied across the numerators and denominators. One student teacher cross-added twice and ended up with the solution presented in Figure 6. In the solution, the participant added across the common denominators, which he/she did with the previous addition task as well.

$$\frac{4}{5}+\frac{2}{3}=\frac{4+3}{5+2}+\frac{2+5}{3+4}=\frac{7}{7}+\frac{7}{7}=\frac{14}{14}$$

Figure 6. An incorrect solution for addition (participant 40)

One participant seemed to demonstrate uncertainty when presenting two alternative solutions. The other solution procedure and the resulting answer $1\frac{7}{15}$ were correct, but he/she had marked the following method as the correct one: $\frac{4}{5}+\frac{2}{3}=\frac{4+3}{5+3}+\frac{2+5}{3+5}=\frac{7}{8}+\frac{7}{8}=\frac{14}{8}=1\frac{6}{8}$. In general, the participants who made errors with their addition solutions did not seem to notice that their answers were unreasonable. For example, when adding $\frac{4}{5}+\frac{2}{3}$, it is not possible to get $\frac{1}{5}$ as an answer because it is smaller than $\frac{4}{5}$. The number of different incorrect solution methods in this task may indicate that when the participants did not remember or understand the operation procedure they seemed to guess an algorithm to use for the solution.

Subtraction with different denominators: $\frac{3}{4}-\frac{1}{2}$. This task was correctly performed by 47 participants (80%); four of them used pie charts to present their solutions while the others showed their solutions with some mathematical steps. One participant converted the fractions first to percent and then after calculating the answer it was converted to the correct fraction form, which was an example of using different representation forms to find the solution. Moreover, two participants used decimals, and one of them arrived at a right decimal form answer. Three participants left this task blank.

Several participants made technical E2 writing errors also with this task. Procedural E5 errors were made as well, and the most common of them was the use of unnecessarily large common denominators: seventeen participants multiplied both fractions in order to get 8 as the common denominator. This may indicate a lack of number sense related to whole numbers or a poor understanding of the subtraction operation since it was not necessary to multiply both fractions since the denominators were 4 and 2. Moreover, one participant found the common denominator 8 but kept multiplying the numerators following the same logic as he/she did in the latter addition task as well. Two student teachers who added across in addition used a similar method here as well. Thus, they subtracted across the numerators and denominators and wrote the problem out as: $\frac{3}{4}-\frac{1}{2}=\frac{2}{2}=1$. Here, again, it can be seen that the participants did not seem to notice that it was impossible to give 1 as a reasonable answer.

Subtraction with a whole number: $1-\frac{2}{6}$. Unlike the first subtraction task, only 27 participants (less than 50%) gave the correct answer for this task. However, the most common error (E1) occurred when 29 participants left their answer as $\frac{4}{6}$ without simplifying it. Thus, most of the participants were able to work through the subtraction procedure, but they did not present the answer in such a form, which was defined as correct in this study. One student teacher simplified the fraction first from $\frac{2}{6}$ to $\frac{1}{3}$, but after subtracting $1-\frac{1}{3}$, he/she gave the answer in decimal form (0.666). Two participants used colored circles, and one of them arrived at the correct answer. One participant left the task blank.

Mathematical writing errors E2 were also common with this task; ten participants used mathematical symbol writing incorrectly, and some had missing steps in their solutions. Moreover, three participants used a procedurally correct but an unnecessarily long solution method (E5): $1-\frac{2}{6}=\frac{1}{1}-\frac{2}{6}=\frac{1\cdot 6}{1\cdot 6}-\frac{2\cdot 1}{6\cdot 1}=\frac{6}{6}-\frac{2}{6}=\frac{4}{6}=\frac{2}{3}$. After converting the whole number 1 to a fraction form, they multiplied both fractions to get 6 as a common denominator, even though there was no need to multiply the latter fraction by 1. This seemed inefficient, and the participants seemed to do this routinely without thinking about the meaning of multiplying by 1.

Multiplication with different denominators: $\frac{3}{4}\cdot\frac{2}{5}$. Only 22 student teachers (37%) gave the correct answer by showing some mathematical steps in this task. Three participants used decimals, but they arrived at three different incorrect answers. Ten students left this task blank, which may indicate that they were more uncertain with multiplication than with the operations in the previous tasks.

The difficulty with the multiplication operation was seen also with the number of participants making procedural E6 errors. Eleven participants cross-multiplied the numerators and denominators, which they did in two different ways: $\frac{3}{4}\cdot\frac{2}{5}=\frac{4\cdot 2}{3\cdot 5}=\frac{8}{15}$ or $\frac{3}{4}\cdot\frac{2}{5}=\frac{3\cdot 5}{4\cdot 2}=\frac{15}{8}=1\frac{7}{8}$. Interestingly, one participant used a correct multiplication algorithm first but then crossed it out and used the latter of the faulty methods presented in previous the example.

Another E6 error in the multiplication operation was the use of common denominators, even though this was unnecessary. Altogether, seven participants multiplied both fractions to get 20 as the common denominator. One of them gave $\frac{120}{400}$ as an answer; the others kept 20 as the denominator after multiplying the numerators and arrived at a procedure as follows: $\frac{3}{4}\cdot\frac{2}{5}=\frac{3\cdot 5}{4\cdot 5}\cdot\frac{2\cdot 4}{5\cdot 4}=\frac{15}{20}\cdot\frac{8}{20}=\frac{120}{20}=\frac{60}{10}=6$. Again, the participants seemed to be uncertain about the role and use of denominators, and they did not notice that a whole number solution was an impossible answer for this task.

Interestingly, none of the participants who correctly solved the multiplication $\frac{3}{4}\cdot\frac{2}{5}$ used the option of simplifying the numbers 2 and 4 before multiplying across the numerators and denominators. This can be interpreted as a rote understanding of the algorithm or a limited number sense when seeing multiple numbers.

Division by a whole number: $\frac{3}{4}$ / 3. Similar to the results in multiplication, 22 participants gave the correct answer for the division task. Six of them used the mathematical invert-and-multiply procedure and showed the steps that led to the correct solution. Two participants first converted the divisor 3 to fraction form and then wrote the correct answer. However, it was not possible to find out whether they followed the correct division procedure or whether they just divided across since they wrote as follows: $\frac{3}{4}/\,3=\frac{3}{4}\;/\;\frac{3}{1}=\frac{1}{4}$. Moreover, five participants used decimals in their solutions; two of them arrived at the correct answer in fraction form and one gave a right answer as decimals. Like in the previous tasks, the participants using pictures were more successful in finding the correct answer than those who used mathematical symbol representations but made errors in them. However, it was difficult to find out the mathematical thinking model behind the correct answer in these pictorial representations as well. For example, it is unclear whether the answer in Figure 7 refers to one of the colored parts in the rectangle or to the remaining white part.

Figure 7. A pictorial solution for the division task (participant 11)

In general, solving the fraction division task by showing their solution steps seemed challenging for the student teachers. A total of eighteen participants made mathematical writing errors (E2), and similar to the multiplication task, ten participants left the division task blank; four of them did this in the case of multiplication as well. In addition to these technical errors, even six different error subtypes that were made altogether by twenty participants were found for the division operation. The most common of these procedural E7 errors occurred when the whole number divisor 3 was converted to fraction form. Some participants seemed to prefer having the same denominators for both the dividend and divisor even though it was unnecessary, and thus, eight of them converted the divisor to $\frac{12}{4}$ and one incorrectly to $\frac{4}{4}$; four participants also changed the divisor 3 to the form $\frac{3}{3}$. Interestingly, only two of those who used the form $\frac{12}{4}$ went further in their solutions but they arrived at the different incorrect answers presented in Figure 8.

$$\frac{3}{4}/3 = \frac{3}{4}/\frac{12}{4} = \frac{4}{4} = 1 \qquad \frac{3}{4}/3 = \frac{3}{4}/\frac{12}{4} = \frac{3}{4}\cdot\frac{4}{12} = \frac{12}{48} = 3$$

Figure 8. Incorrect solutions for division (participants 15 and 49)

As can be seen in the examples above, the participants made multiple errors in their solutions; in the example on the left, the student teacher has obviously divided the numerator 3 by 12 and kept the denominators to get $\frac{4}{4}$, whereas the other student teacher seems to use the invert-and-multiply procedure, but then incorrectly divides 48 by 12. Other procedural errors for the division operation were (a) dividing the numerator or both the numerator and denominator by the whole number divisor, (b) first multiplying the numerator and denominator by the divisor and then dividing the new fraction by it, (c) dividing across by a fraction form divisor, and (d) cross-multiplying by the inverted divisor. Similar to addition with different denominators, the number of different incorrect solution methods in the division task seems to indicate that the participants are guessing the solution methods when they do not remember or understand the correct algorithm; some participants even wrote on the research questionnaire that they did not remember how to divide fractions.

In this section, the participating student teachers' solutions for fraction tasks were described in general and in terms of their errors and difficulties with the six routine fractions tasks. The analysis revealed several limitations in their CCK on fractions and also some other limitations in their basic knowledge of mathematics; these findings were not directly connected to their knowledge of fractions. In the next section, the most important results of this study will be summarized and discussed.

6 Discussion and conclusions

In this study, student teachers' CCK on fractions was investigated by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. Many of the findings concerning their procedural errors in fraction operations are in line with findings in previous studies (e.g., Newton, 2008; Van Steenbrugge et al., 2014; Young & Zientek, 2011). In other words, the participants in this study had difficulties with all fraction operations and especially with division and multiplication. Many of them seemed to have a rule-based and rote understanding of the algorithms, and they used several incorrect methods for their solutions. Moreover, they seemed to lack knowledge of using other representations when not being able to use a correct algorithm. It was also seen in this study that student teachers have difficulties in using fraction number sense.

Different problems concerning the teaching and learning of fractions have been reported for decades, and the need to develop student teachers' knowledge of fractions has also been reported earlier (e.g., Van Steenbrugge et al., 2014). This study is consistent with the previous findings about student teachers' limited CCK of fractions. In addition, the study reveals some other limitations in their mathematical CCK.

In general, it was surprising that so many of the participating student teachers made several types of errors and that there was so wide difference between the participants when solving the fraction tasks. The participants were expected to be familiar with the routine tasks and the fraction content included in the tasks, since they had recalled and repeated this content in their previous mathematics course in teacher education. The uncertainty that many participants demonstrated in their CCK was seen in the number of tasks left blank and, for example, in their lack of using different fraction forms coherently throughout the solutions. Moreover, showing how to solve a routine task step-by-step seemed to be challenging for most of the student teachers; the more steps needed to find a solution, the more difficult it became to write out the procedures and the more errors the participants made. Like student teachers in Jakobsen et al.'s study (2014), many participants used in their solutions incorrect mathematical notations and moreover, they used separate solution steps that formed illogical statements without constructing a logical solution procedure.

The participants in this study also demonstrated limitations in their basic knowledge concerning mathematical symbol writing and the use of different representation forms. This is an important finding since these errors did not seem to be directly connected to fractions but rather they seemed to be general limitations in student teachers' CCK, which may have an effect when student teachers work with fraction as well. For example, some of the student teachers were misusing the equal sign, and they made errors in differentiating the symbols to simplify a fraction and to divide it. Making this kind of errors in their mathematics teaching might be confusing for elementary school students. Unlike Newton's study (2008), where none of the 85 participants used pictures to solve routine fraction tasks, seven participants in the present study used pictures to find the correct answers. However, it seemed that pictorial representations were used with tasks where the participants were uncertain about the correct algorithm, and many of the pictures that they presented could be seen as they mental images of the fractions and not as representations of the solution procedures needed for the tasks. As Moss et al. (1999) have stated, especially the use of pie charts may be misleading in elementary mathematics teaching. Thus, it seems that the becoming teachers need to learn how to better use pictorial representations to visualize abstract mathematical procedures. Moreover, a robust knowledge of correct mathematical algorithms is needed as well since pictorial illustrations with simple fractions such as $\frac{3}{4}$ and $\frac{1}{2}$ work well, but the use of pictures becomes complicated for fractions like $\frac{13}{41}$ and $\frac{11}{21}$. Some participants in this study used also decimals throughout the fraction tasks but they did not seem to notice the errors that occurred in their solutions when they converted improper fractions to decimals (c.f. Muir & Livy, 2012).

Moreover, many student teachers in this study did not notice their incorrect statements and unreasonable answers even in the simplest cases. However, determining equivalence and judging the reasonability of answers are essential parts of fraction number sense (Lamon, 2020) and CCK for mathematics teachers in their daily work (Ball et al., 2008). This finding like the previous one concerning mathematical symbol writing and using different representation forms may not be connected to fraction tasks only and should therefore be researched further.

Further, an interesting finding was that the participating student teachers seemed to guess at which algorithm to use when they did not remember or understand the correct solution method. Often, they seemed to remember some separate steps of the algorithms instead of understanding the procedures as a whole. Also, as Newton (2008) states, it seems that even though student teachers remember many procedures, they use them in inappropriate ways with fractions. For a mathematics teacher, a robust CCK goes beyond rote learning and memorization of algorithms since “teaching requires knowledge beyond that being taught to students” (Ball et al., 2008, p. 400).

Although student teachers do not need to hold a level of expertise equivalent to that of an experienced elementary mathematics teachers, they should not be regarded as novices in their mathematical CCK. However, student teachers may enter their studies in teacher education with different prior mathematical knowledge and with different kinds of experiences in mathematics teaching and learning. As seen in this study and in previous research (e.g. Newton, 2008), not all student teachers are competent in their basic knowledge of fractions, and the limitations found in their CCK may not predict success in teaching of fractions in their future profession as elementary mathematics teachers (Van Steenbrugge et al., 2014). Thus, teacher educators need to pay attention to student teachers’ individual differences and to be aware of their different error patterns (Young & Zientek, 2011). Especially, the results in this study reveal that student teachers need a deep knowledge of fractions and mathematical symbol writing and the meaning of the procedures as well; it is not enough to be able to produce correct answers for mathematical tasks. To enhance this knowledge and student teachers' ability to interpret others' mathematical solutions as well student teachers should be given fraction tasks to be solved in different ways like Jakobsen et al. (2014) and Maciejewski and Star (2016) conclude in their studies.

The present study, conducted in the Swedish context, confirms the results from other countries during recent decades. Thus, it can be stated that there is still much to do when developing student teachers' CCK on fractions and other mathematical content as well. Since the present study concerned only a group of student teachers in one Swedish university, a limitation of the study is the inability to generalize the results beyond this population. However, some errors did occur across the participants, and this may rise questions about general difficulties in student teachers' CCK. For example, student teachers' use of mathematical symbol writing and mathematical representations for topics other than fractions could be addressed in further research. Moreover, maybe the biggest challenge in teacher education is how to address student teachers' individual differences and their various difficulties in mathematics.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449–466. https://doi.org/10.1086/461626

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194–222. https://doi.org/10.2307/749118

Charalambous, C. Y., Hill, H. C., Chin, M. J., & McGinn, D. (2020). Mathematical content knowledge and knowledge for teaching: exploring their distinguishability and contribution to student learning. J*ournal of Mathematics Teacher Education*, *23*(6), 579–613. https://doi.org/10.1007/s10857-019-09443-2

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics*, *64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111–144. https://doi.org/10.2307/749646

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: a model. *Journal of Education for Teaching*, *15*(1), 13–33. https://doi.org/10.1080/0260747890150102

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed). *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Lawrence Erlbaum.

Hoover, M., Mosvold, R., Ball, D. L., & Lai, Y. (2016). Making progress on mathematical knowledge for teaching. *The Mathematics Enthusiast*, *13*(1 & 2), 3–34. https://doi.org/10.54870/1551-3440.1363

Häkkinen, K., Tossavainen, T., & Tossavainen, A. (2011). Kokemuksia luokanopettajaksi pyrkivien matematiikan soveltuvuustestistä Savonlinnan opettajankoulutuslaitoksessa. In E. Pehkonen (Ed.), *Luokanopettajaopiskelijoiden matematiikkataidoista*, Tutkimuksia 328 (pp. 47–64). Department of Applied Educational Science, University of Helsinki.

Jakobsen, A., Ribeiro, C. M., & Mellone, M. (2014). Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task. *Nordic Studies in Mathematics Education, 19*(3-4)*,* 135–150.

Jóhannsdóttir, B., & Gísladóttir, B. (2014). Exploring the mathematical knowledge of prospective elementary teachers in Iceland using the MKT measures. *Nordic Studies in Mathematics Education, 19*(3-4), 21–40.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Lawrence Erlbaum Associates.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Information Age.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (Fourth Ed.). Routledge.

Lin, C.-Y., Becker, J., Byun, M.-R., Yang, D.-C., & Huang, T.-W. (2013). Preservice teachers' conceptual and procedural knowledge of fraction operations: A comparative study of the United States and Taiwan. *School Science and Mathematics*, *113(1)*, 41–51. https://doi.org/10.1111/j.1949-8594.2012.00173.x

Löwing, M. (2016). *Diamant – diagnoser i matematik: Ett kartläggningsmaterial baserat på didaktisk ämnesanalys* [Doctoral dissertation, University of Gothenburg]. http://hdl.handle.net/2077/47607

Ma, L. (2010). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. *Anniversary Edition*. Routledge.

Maciejewski, W., & Star, J. R. (2016). Developing flexible procedural knowledge in undergraduate calculus. *Research in Mathematics Education*, *18*(3), 299–316. https://doi.org/10.1080/14794802.2016.1148626

Marchionda, H. (2006). *Preservice teachers' procedural and conceptual understanding of fractions and the effects of inquiry-based learning on this understanding* [Doctoral dissertation, Clemson University]. https://tigerprints.clemson.edu/all_dissertations/37

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122–147. https://doi.org/10.2307/749607

Muir, T., & Livy, S. (2012). What do they know? A comparison of pre-service teachers' and in-service teachers' decimal mathematical content knowledge. *International Journal for Mathematics Teaching and Learning*, *2012, December 5th*, 1–15. Retrieved from http://www.cimt.org.uk/journal/muir2.pdf

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, *45*(4), 1080–1110. https://doi.org/10.3102/0002831208320851

Olanoff, D., Lo, J.-J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast*, *11*(2), 267–310. https://doi.org/10.54870/1551-3440.1304

Radatz, H. (1979). Error analysis in mathematics education. *Journal for Research in Mathematics Education*, *10*(3), 163–172. https://doi.org/10.2307/748804

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Skolverket [Swedish National Agency for Education]. (2011). *Curriculum for the compulsory school, preschool class and school-age educare. Revised 2018*. Skolverket.

Skolverket [Swedish National Agency for Education]. (2016). *TIMSS 2015. Svenska grundskoleelevers kunskaper i matematik och naturvetenskap i ett internationellt perspektiv. Internationella studier 448*. Skolverket.

Skolverket [Swedish National Agency for Education]. (2019). *PISA 2018. 15-åringars kunskaper i läsförståelse, matematik och naturvetenskap. Internationella studier 487*. Skolverket.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31(1)*, 5–25. https://doi.org/10.2307/749817

Tirosh, D., Fischbein, E., Graeber A. O., & Wilson, J. W. (1998). Prospective elementary teachers' conceptions of rational numbers. Retrieved from http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html

Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education*, *25*(1), 166–175. https://doi.org/10.1016/j.tate.2008.08.003

Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2014). Preservice elementary school teachers' knowledge of fractions: a mirror of students' knowledge? *Journal of Curriculum Studies*, *46*(1), 138–161. https://doi.org/10.1080/00220272.2013.839003

Zhou, Z., Peverly, S.T., & Xin, T. (2006). Knowing and teaching fractions: A cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology*, *31*, 438–457. https://doi.org/10.1016/j.cedpsych.2006.02.001

Young, E., & Zientek, L. R. (2011). Fraction operations: An examination of prospective teachers' errors, confidence, and bias. *Investigations in Mathematics Learning*, *4*(1), 1–23. https://doi.org/10.1080/24727466.2011.11790307