Mathematical thinking and understanding in learning of mathematics

We both editors have wondered and studied "***What is mathematical thinking?***" more than thirty years. At least the question could be recognized behind most of our research projects concerning studies in mathematics education from little school children to university students. The concept "mathematical thinking" can be found in several studies of mathematics education, in national curricula or in media during the decades all over the world. We searched words "mathematical thinking" from the database of international scientific articles, and we found 456 707 mentions at first time. These are the main reasons why we have chosen "mathematical thinking" as the central concept of the special issue. The other interesting question from our point of view is how a student can express his/her mathematical thinking? By answering this question, we have made simple model for the teacher education purposes, and we call it "languaging" (of mathematical thinking). In the following, we lead to the above concepts and prepare the presentation of articles in this journal.

Sternberg (1996) has studied different approaches to the concept of mathematical thinking. He found at least five different points of view to describe the concept. They are anthropological, information process, mathematical, pedagogical, and psychometric approach. For example, in the anthropological approach the central starting point is the surrounding culture (e.g. ethnomathematics d'Ambrosio, 1985), in the information process different types of knowledge in mathematics (e.g. Joutsenlahti, 2009) or in psychometric the abilities in doing mathematics (e.g. Krutetskii, 1976). We can interpret that the pedagogical approach in the school context takes account on beliefs and problem solving (e.g. Pehkonen, 1998, 2007 and Hannula, 2004) in thinking processes. We have used the information process approach in describing the concept of mathematical thinking, and we described knowledge (meaningful information) as conceptual and procedural (Hiebert & Lefevre, 1986). Student's metacognition guides his/her thinking. Oikkonen and Hannula have taken the viewpoint to mathematical thinking David Tall's framework of the three worlds of mathematics in their article "*The three worlds and two sides of mathematics and a visual construction for a continuous nowhere differentiable function*". In their theoretical article, they further elaborate Tall's framework and demonstrate this framework by discussion on the definition of continuity. Kayan Fadlelmula's article "*A PRISMA Systematic Review on Enablers and Obstacles in Teaching and Learning of Mathematics*" is systematic review on the current issues positively and negatively affecting teaching and learning in mathematics and the data was gathered from the studies published in the LUMAT -journal. Metsämuuronen's and Ukkola's article "*Rudimentary stages of the mathematical thinking and proficiency - Mathematical skills of low-performing pupils at the beginning of the first grade*" based on the national-level dataset (n = 7770) at grade 1 of primary school in Finland and the focus is on those pupils whose preconditions are so low that they are below the first measurable level of proficiency in the common framework with reference to mathematics.

When we were young teachers, we often thought ***How we can express mathematical thinking***? and especially how we could encourage students to do it by many ways? Traditionally, in mathematics classes, students work quietly with their own textbook and asked for help only from the teacher. We didn't often know what kind of thoughts our students had about the solutions processes of mathematics problems. Nevertheless, we finally understood that if we get a student to speak about mathematics – we get him/her to think mathematics and we can hear his/her mathematical thinking! Also, we can see it if the student does it by writing or/and drawing (see e.g. Morgan, 2001). We call this process languaging of mathematical thinking, which is based on a model of four "languages". They are mathematical symbolic language, natural language, pictorial language, and tactile action language (Joutsenlahti & Kulju, 2017; Joutsenlahti & Perkkilä, 2019). The most effective benefit of languaging for the student is that when the student expresses mathematical thoughts by his/her own words then he/she structures his/her thinking and by that way understands mathematical concepts and procedures better. It is for the teacher easy to evaluate student's thinking and give help if needed. When a student expresses his/her mathematical thinking (s)he can use different multimodal approaches (e.g. the four "languages"). Theoretically, the multimodal languaging model is related to multiliteracy (Kalanzis & Cope, 2012). When a student makes meanings for the mathematical text the languages can be seen as a multi-semiotic approach, where the different languages make it possible to construct many kinds of meanings for concepts in versatile contexts (Joutsenlahti & Perkkilä, 2019). Björklund's, Ekdahl's, Kullberg's and Reis's article "*Preschoolers' ways of experiencing numbers*" directs attention to 5–6-year-olds' learning of arithmetic skills through a thorough analysis of changes in the children's ways of encountering and experiencing numbers. The aim of Kaitera's and Harmoinen's study was to map whether a teaching approach, which focuses on teaching general heuristics for mathematical problem-solving by providing visual tools called

Problem-solving Keys, would improve students' performance in tasks and skills in justifying their reasoning in their article "*Developing mathematical problem-solving skills in primary school by using visual representations on heuristics*". Francisco's article "*Supporting Argumentation in Mathematics Classrooms: The Role of Teachers' Mathematical Knowledge*" addresses a documented need for a better understanding of the relationship between mathematical knowledge for teaching and instruction by focusing on how the knowledge influences teachers' support of argumentation. Rinneheimo concentrates on the use of languaging exercises in the engineering mathematics course in Finland in her article "*I solved a derivative – but what does it actually mean? Languaging and conceptual understanding in engineering mathematics.*"

We have thought about the relationship between **conceptual understanding and mathematical thinking**. The development of mathematical thinking is emphasized in the Finnish curricula of pre-school and school education. The main goal of the curricula is to develop a student's mathematical thinking and understanding about mathematics. Hiebert and Lefevre (1986, p. 3—8) have defined conceptual and procedural mathematical knowledge. According to them, procedural knowledge refers to those procedures that are needed to solve mathematical tasks and problems. Conceptual knowledge can be described as the richness of knowledge in relationships between things which 'can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions, so that all pieces of information are linked to some network' (Hiebert & Lefevre, 1986, pp 3—4). Both definitions of procedural and conceptual knowledge share commonalities with Skemp's definitions of similar concepts (Skemp, 1976). Hiebert and Lefevre's (1986) description of procedural knowledge resembles the definition of instrumental knowledge by Skemp (1976), which can be seen as the application of finished formulas and models to certain kinds of tasks. In the definitions of conceptual knowledge, both Hiebert and Lefevre (1986) and Skemp emphasize understanding about the connections made by mathematical concepts. When these connections between concepts are built purposefully in teaching, students gradually develop an understanding about the network of mathematical concepts. Thus, a student does not use loose mathematical concepts; (s)he understands the whole system of them. The interactivity of the learning environment, student's timely support and received feedback and the process of becoming accepted as oneself contributes to the construction of a sustainable first-hand mathematical knowledge and skills base, i.e. the building of mathematical competence. It is important that the learning community (teachers and students) have a feeling let's do it together, talk and model mathematical solutions through the means of languaging.

Early mathematical skills build a foundation for the individual's comprehending learning of school mathematics skills and mathematical knowledge. The level of development and mathematical knowledge of students' early mathematical skills meet no later than preschool. In order to develop the student's mathematical thinking skills, we need to understand how (s)he learns mathematics. Preschool age and elementary school students are on the concrete level of mathematical thinking, and it is reflected in their actions. Conceptual understanding develops best in a sociocultural context by collaborative working methods where students construct their own mathematical thinking through drawings, using mathematical symbolic language, concrete and verbal actions (e.g. Perkkilä & Joutsenlahti, 2021). This viewpoint is in line with Vygotsky's theory, which emphasizes the sociocultural perspective. Building a math-speaking community where everyone is a teacher and learner is crucial for students building a conceptual network in a particular math area. (e.g. Fuson, 2019.) This allows all students from preschool to university to build their own mathematical thinking from their own mathematical skill level. By supporting the construction of student's mathematical thinking and conceptual understanding, we support sustainable development from the perspective of learning mathematics (e.g. Joutsenlahti & Perkkilä, 2019; Perkkilä & Joutsenlahti, 2021).

Algebraic thinking is an important part of mathematical thinking. Both Sanna Wettergren and Inger Eriksson and Natalia Tabachnikova have studied how to promote young students' algebraic thinking in their articles. Wettergren explored how teaching aiming to promote young students' algebraic thinking can be designed in her article "*Identifying and promoting young students' early algebraic thinking*". Eriksson and Tabachnikova have sought answers for the development of algebraic thinking with an example based on a case study that describes how young students can theoretically study and reflect some aspects of the equations in their article "*IE "Learning models": utilising young students' algebraic understanding of equations*". Kambara's and Tossavainen's articles focus on examining conceptual understanding in students studying to be a teacher. Kambara's article "*Understanding of "proportion" and mathematical identity: A study of Japanese elementary school teachers*" explores and clarifies the level of conceptual understanding of "proportions" among

Japanese students who hope to become elementary school teachers in the future. Tossavainen's article "*Student Teachers' Common Content Knowledge for Solving Routine Fraction Tasks*" focuses on the knowledge base that Swedish elementary student teachers demonstrate in their solutions for six routine fraction tasks.

We think that we have got very good sample of scientific articles to our Special Issue. Thank you for all the writers, you have done excellent work!

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Guest Editors
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References

Fuson, K. C. (2019) Relating math words, visual images, and math symbols for understanding and competence. *International Journal of Disability, Development and Education*, 66(2), 119–132. https://doi.org/10.1080/1034912X.2018.1535109

Hannula, M. S. (2004). *Affect in mathematical thinking and learning*. [Doctoral thesis, University of Turku]

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Eds.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 1–27). Lawrence Erlbaum.

d'Ambrosio, U. (1985). Ethnomathematics and Its Place in the History and Pedagogy of Mathematics. *For the Learning of Mathematics*, *5*(1), 44–48. http://www.jstor.org/stable/40247876

Joutsenlahti J. (2009). The features of mathematical thinking among Finnish students in upper-secondary school. Teoksessa C. Winslow (Eds.), *Nordic Research in Mathematics Education: proceedings from NORMA08 in Copenhagen, April 21-April 25, 2008* (pp. 315–320). Sense Publishers.

Joutsenlahti, J., & Kulju, P. (2017). Multimodal Languaging as a Pedagogical Model - A Case Study of the Concept of Division in School Mathematics. *Education Sciences, 7*(1), 9. https://doi.org/10.3390/educsci7010009

Joutsenlahti, J. & Perkkilä, P. (2019). "Sustainability Development in Mathematics Education—A Case Study of What Kind of Meanings Do Prospective Class Teachers Find for the Mathematical Symbol "2/3"?" *Sustainability* 11(2), 457. https://doi.org/10.3390/su11020457

Kalantzis, M. & Cope, B. (2012). *Literacies*. Cambridge University Press.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. The University of Chicago Press.

Morgan, C. (2001). The place of pupil writing in learning, teaching and assessing mathematics. In P. Gates (Ed.) *Issues in Mathematics Teaching*. Routledge, 234–44.

Pehkonen, E. (1998). On the concept 'mathematical belief'. In E. Pehkonen & G. Törner (Eds.), *The state-of-art in mathematics-related belief research. Results of the MAVI activities* (Research Report 195, pp. 37–72). University of Helsinki.

Pehkonen, E., Björkqvist, O., & Hannula, M. (2007). Problem solving as a teaching method in mathematics education. In E. Pehkonen, M. Ahtee & J. Lavonen (Eds.), *How Finns learn mathematics and science* (pp. 121–131). Sense publishers.

Perkkilä, P. & Joutsenlahti, J. (2021). Academic Literacy Supporting Sustainability for Mathematics Education - A Case: Collaborative Working as a Meaning Making for "2/3"? In E. Jeronen (Ed.), *Transitioning to Quality Education* (Transitioning to Sustainability 4, pp. 163–188). MDPI. https://doi.org/10.3390/books978-3-03897-893-0-8

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20–26.

Sternberg, R. (1996). What is mathematical thinking? In R. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 303–318). Lawrence Erlbaum.