Beliefs-oriented subject-matter didactics: Design of a seminar and a book on calculus education

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This paper presents a modified approach to subject-matter didactics, in which the focus is not on the content itself, but on the students' view of the content. The introduction deals with an overview of subject-matter didactics and the notion of beliefs used in this paper. The main portion of the paper deals with presenting the concepts of a book and a seminar based on the student-centered subject-matter didactics approach. For the first qualitative evaluation, selected reflections of students are analyzed. Finally, initial findings are summarized and an outlook is provided.

Keywords: belief systems, calculus education, pre-service teacher training, subject-matter didactics

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1 Introduction

The analysis of mathematical content and contexts as well as their translation into practical concepts for teaching mathematics at school are central tasks of mathematics education research. In German-speaking countries, there is a long tradition in this field of research under the title "Stoffdidaktik" or "Subject-Matter Didactics" (cf. Hefendehl-Hebeker, 2016), which still has a strong influence on research and teacher education today:

> Stoffdidaktik has been a dominant approach to mathematics education research within the German speaking countries, which puts the analysis of the mathematical subject matter at its heart. It has been the prominent approach to research until the 1980s. Nowadays, it still influences research in mathematics education in German speaking countries. (Hußmann et al., 2016, p. 1-2)

The focus of classical subject-matter didactics is the mathematical content taught at school. The aim is to provide students and teachers with an accessible approach to mathematical content knowledge. For this purpose, the subject-matter didactics investigate:

- "Essential concepts, procedures and relationships including appropriate formulations, illustrations and arrangements for teaching
- Essential structures and domain-specific ways of thinking

- The inner network of paths by which the components are connected and possible learning paths throughout the domain" (Hefendehl-Hebeker et al., 2019, p. 26).

In this paper, a slightly modified and novel approach to subject-matter didactics is described and applied. Instead of assuming a fixed mathematical framework, which is "elementarized" for school, the focus lies on different ways of thinking about mathematical concepts, disciplines, and mathematics in general. Theoretical approaches and empirical findings in the context of mathematics-related beliefs form the basis of learner-centered subject-matter didactics as will be described in the following sections, using calculus education as an example. The authors present the design of a seminar and a book on calculus education using this approach. Furthermore, a brief insight into the evaluation of the seminar and the book at the University of Siegen in the summer of 2020 is provided.

2 The underlying notion of beliefs

The general idea for the conceptualization of the textbook and the seminar on calculus education lies in the concept of belief and its application in calculus. There are many definitions of the term "belief" (cf. Thompson, 1992) and related terms (cf. Pajares, 1992), which differ considerably. In our conceptualization, we rely on the well-known definition of Schoenfeld (1985), who considered beliefs as mental structures that determine the behavior of a person:

> Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics and control operate. (Schoenfeld, 1985, p. 45)

Hence, Schoenfeld (1985) defined the term "belief system" in relation to a person's behavior when dealing with mathematical problems. In particular, Schoenfeld applied the term to the description of problem-solving situations. However, the concept can be applied to mathematical knowledge development processes in general. According to Schoenfeld, beliefs and belief systems are mental structures with cognitive and affective components (cf. Schoenfeld, 1985, 1992), which substantially determine behavior in addition to other important factors (resources, heuristics, and control).

The decision to make mathematics-related beliefs the basis of a subject-matter-oriented seminar and a book can be justified by the idea that those beliefs have a considerable impact on students' learning of mathematics. This idea is also described in the well-known quote by Goldin et al. (2009):

> To sum up, *beliefs matter*. Their influence ranges from the individual mathematical learner and problem solver and the classroom teacher, to the success or failure of massive curricular reform efforts across entire countries. (p. 14, emphasis in the original)

The development of adequate beliefs about mathematics by prospective teachers can be understood as a critical goal of the teacher-training program, subsequently affecting teaching at school and, in turn, also the development of beliefs by students in mathematics classes:

> Because attitudes are acquired in learning processes in which the (social) environmental conditions have a substantial influence, it can also be argued that the attitudes of teachers have a substantial influence on the attitudes of students — on one hand, in direct communication and interaction in a mathematics class, and on the other hand, indirectly through the concrete design (choice of material and methods, and assessment system) of a mathematics class. (Grigutsch et al., 1998, p. 4, authors‘ translation)

3 Beliefs for enabling perspectives on calculus

In this work, we are particularly interested in domain-specific beliefs (i.e., those that refer to a specific mathematical domain — in our case, calculus). The basis for our work can be traced back to the article "Domain-Specific Beliefs of School Calculus" by Witzke and Spies (2016). In this article, the authors discussed, among other things, that some domain-specific beliefs in school calculus are more dominant than others, according to the way calculus is taught at school and the way students receive and construct the knowledge. Based on a qualitative content analysis with the inductive specification of deductive categories, Witzke and Spies (2016) identified six deductive categories (for examples of the categories from the data, see Witzke & Spies, 2016, p. 144):

- *Logical-structural orientation*, which focuses on deduction and proof, as well as the understanding of (intra-mathematical) connections among concepts and their underlying structures.

- *Abstract-terminological orientation*, which focuses on formal rigor, the use of precise mathematical language, and the understanding of mathematical objects as abstract entities.
- *Toolbox orientation*, where operating calculus performs certain rules, formulas, and procedures in a schematic way (e.g., how to determine a derivative, extreme values, or similar).
- *Utility orientation*, which focuses on extra-mathematical applications or mathematical modeling.
- *Empirical orientation*, in which objects related to the real world and basic concepts derived from these perceptions are the focus.
- *Symbolical orientation*, in which objects of calculus are identified with characteristic symbols.

4 Structure of the seminar and the book

Various German books on calculus education (German: Didaktik der Analysis) in the tradition of subject-matter didactics are available. Classics include, for example, "Analysis verständlich unterrichten" (English: Teaching Calculus in a Comprehensible Way) by Rainer Danckwerts and Dankwart Vogel (see Danckwerts & Vogler, 2006) or "Didaktik der Analysis" by Werner Blum and Günter Törner (see Blum & Törner, 1983). However, there are also more recent books such as "Didaktik der Analysis: Aspekte und Grundvorstellungen zentraler Begriffe" (English: Calculus Education: Aspects and Basic Ideas of Essential Concepts) by Gilbert Greefrath, Reinhard Oldenburg, Hans-Steffan Siller, Volker Ulm, and Hans-Georg Weigand (see Greefrath et al., 2016).

All these books take different perspectives on calculus and the teaching of calculus (e.g., a focus on extra-mathematical application). Generally, their structures follow the systematic structure of calculus (i.e., start with basic concepts such as sequences and series, and afterward discuss functions, derivatives, and integrals, as well as their adequate introduction depending on the authors' perspectives). The seminar and the textbook of calculus education presented in this paper take a different approach. The content is structured in terms of typical belief systems of mathematics according to different educational perspectives on calculus, namely, the formal-abstract perspective, the empirical-concrete perspective, the "toolbox" perspective, and the application perspective. These four perspectives are the result of a reduction of the orientations mentioned in the study by Witzke and Spies (2016) — the logical-structural orientation and the abstract-terminological orientation merge into the formal-abstract perspective, and the toolbox orientation and the symbolic orientation form the "toolbox" perspective.

This is a subjective selection from the perspective of the authors of this paper, based on their experience with calculus teaching. The decision to use the term "perspectives" instead of "beliefs" or "orientations" in the structure of the textbook is due to the fact that perspectives are actively taken, whereas beliefs or orientations do not necessarily become explicit for the belief-bearers, but possibly represent a "hidden variable" (Goldin et al., 2009) for them. This reinterpretation enables beliefs to be addressed proactively and productively for mathematical teaching and learning. Each of the four perspectives forms a chapter of the textbook or a unit of the seminar, which connects the basic mathematical concepts of calculus and interprets their educational implications according to the focused perspective. In addition, a concluding chapter interconnects the different perspectives on calculus according to a higher point of view gained in the course of the book and the seminar.

The aim is to experience that there is a multitude of views on mathematics or perspectives on mathematics in general and on calculus in particular. The higher point of view is characterized by the knowledge of the manifold of these perspectives and the ability to address adequate belief systems in different contexts of application (a school, university, teacher, or learner) or to adopt and further develop them. An argument for such a conception, under the condition that the students are already familiar with calculus, is offered by the following quotation from "Elementarmathematik vom höheren Standpunkt" (English: Elementary Mathematics from a Higher Standpoint) by Felix Klein:

> I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the most important disciplines of mathematics. I shall often have to talk of problems of algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. [...] In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you here as vivid stimuli for your teaching. (Klein, 2016, p. 1f.)

Hence, the book and the seminar are based on the assumption that a major goal of teaching calculus should be the stimulation of multiple perspectives (see also Green, 1971 on the formation of beliefs as a goal of mathematics education), whereas the formal-abstract perspective does not necessarily have to be taken by students at school. Research on the transition from school to university in mathematics shows that the change of understanding from an empirical-concrete to a formal-abstract belief system of mathematics is associated with major challenges on an epistemological level (cf. Stoffels, 2020; Tall, 2013). To prepare students for university, teachers should nevertheless be aware of this perspective so that teaching does not obstruct this change in the belief system. This can be achieved, for example, by focusing on mathematical activities that are characteristic of the empirical-concrete as well as the formal-abstract belief systems, such as deductive reasoning or the use of symbolic calculations, which are independent of whether mathematics is understood as an ontologically bound empirical discipline or as an abstract formalistic science.

As an introduction to the topic, the first chapter of the book or the first unit of the seminar explicitly addresses the concept of belief on the basis of varied research literature (i.e., Grigutsch et al., 1998; Schoenfeld, 1985) and illustrates it with original examples. To approach one's own belief system of "school calculus" and to reflect on it while working with the book or in the seminar, several stimulating questions are provided:

1. Why should calculus be taught in school?
2. What are typical activities that you associate with calculus at school?
3. What are typical topics that you associate with calculus at school?
4. When do you consider a statement in calculus to be verified?
5. Why should you, as a prospective mathematics teacher, attend a lecture on calculus during your studies for a teaching profession?
6. What are typical mathematical activities that you associate with calculus at university?
7. What are typical topics that you associate with calculus at university?

Finally, the first chapter also presents the normative goals of teaching calculus with reference to German curricula (Conference of the German Ministers of Education and Cultural Affairs, 2015) and the domain-specific educational literature (e.g., Dankwerts & Vogler, 2006; Greefrath et al., 2016).

In the subsequent four chapters, the formal-abstract perspective, the empirical-concrete perspective, the "toolbox" perspective, and the application perspective are initially discussed separately. In this context, the following topics (and others) are addressed:

- Formal-Abstract Perspective:

    - Formal elements in school calculus
    - Real numbers and functions
    - Central theorems of calculus
    - Theorems of calculus at school
- Empirical-Concrete Perspective:
    - Visual representations in the teaching of calculus
    - Empirical belief systems in the history of calculus
    - Geometric representations of theorems in calculus
    - Illustrative approaches to digital technologies
    - Tools and tactile models in calculus teaching
- “Toolbox” Perspective:
    - Algorithms in calculus education
    - Extreme values and other characteristics (“Funktionsuntersuchung”)
    - Determination of functions
    - Determination of extreme values
    - Rules of derivation and integration
- Application Perspective:
    - Interdisciplinary mathematics teaching
    - Modeling with functions
    - Applications from the natural sciences
    - Applications from economics

Certainly, by limiting the sections to one perspective, one loses the rich linking of perspectives. In school calculus, such changes of perspective should also be attempted, as long as they are consciously and intentionally stimulated by the teacher or reflected by the students. To support the multi-perspective view of the students, who are expected to have already experienced the interconnection of the different perspectives in the context of their own calculus education, the authors consciously decided to initially present separate perspectives on calculus — and present them in an integrated way on a meta-level in the last chapter. Thus, in the final chapter, findings on concept development in the context of calculus are discussed. These findings refer to the concept of "Grundvorstellungen" (Vom Hofe & Blum, 2016) as well as subjective domains of experience (SDE) (cf. Bauersfeld, 1983) to describe overarching and interconnecting perspectives on school calculus.

5 Evaluation of the seminar and the book

The book described in this article is based on a script for a seminar on calculus education by Frederik Dilling and Ingo Witzke from 2018. They tested for the first time at the University of Siegen the approach of using belief systems as a basis for studying calculus. In the following three semesters, Frederik Dilling and Ingo Witzke together with Gero Stoffels conducted the seminar with this concept again. The experience from these four semesters was used to further develop the book as well as the underlying concept and to adapt it to the requirements of university students as the main audience.

Three of the semesters took place during the COVID pandemic, so the courses were arranged in a distance-learning format. For this reason, a reading course was conducted with a two-week interval for reading one chapter of the book, completing selected exercises from this chapter, and finally obtaining written feedback from the lecturers. In addition, the topics of the chapters were discussed in groups through videoconferences on several dates.

The following brief insight into the evaluation of the course and the book is based on the summer 2020 semester — the second time the course was conducted. In total, 10 bachelor's students of teaching mathematics for "Gymnasium" (high school) participated in the course. The authors of this paper conducted a hermeneutic descriptive analysis of selected answers to the exercises in the book as well as detailed written feedback on the content and structure of the chapters provided by the students. In this article, only a small glimpse into the data can be given as this is not a comprehensive empirical study.

In the first chapter of the book, the students were asked seven reflective questions to reflect on their own beliefs about calculus (see above). Among other things, they were asked to consider why calculus should be taught at school and university as well as what typical activities and topics they associate with calculus at school and university. Overall, the answers of the participating students to the question of why calculus should be taught at school mostly referred to the applications of calculus in daily life, future work, or as a prerequisite for university studies with a special focus on STEM disciplines. Only one student focused solely on the "foundational aspect of calculus for mathematics in general," "interconnections between mathematical fields," and fostering "abstract concepts and logical argumentations." The beliefs of the students on why it is necessary for them to learn calculus at university focused on deepening their understanding of calculus and their hopes for an improvement in their future teaching. Typical activities and topics assigned to school calculus were related to the "toolbox" perspective. By contrast, the mentioned university activities and topics can be assigned to the formal-abstract perspective. One student, for example, provided the following reflection on university activities:

> This is closely linked to the topics [of university mathematics]. The 'sitting on problems for a long time' but also talking with others about the exercise sheets. A lot of thinking, not 'understanding the principle and then using a calculator' as in school. Proving that it is also a big topic that is usually completely new to you. (authors' translation)

After this reflection, the students had to work on the well-known isoperimetric problem of maximizing the area of a rectangle to a given fixed circumference. After solving the problem, the students had to reflect if their answers to the previous reflective questions were fit for this task and their solutions. Hence, from the beginning of the book, the authors foster an awareness of the students' own beliefs and valuations. To illustrate this connection, the answer of another student is given, but it is important to mention, that at this moment, the student had a different notion of the formal-abstract perspective than that intended in the book:

> In the first task [these are the initial reflective questions], I wrote that a pupil should learn problem-solving strategies at school and be enabled to relate models to the environment. Hence, in my opinion, the empirical-concrete perspective and the formal-abstract perspective apply and are directly related to the initial task. (authors' translation)

Similar reflecting questions with a focus on the students' beliefs are provided throughout the book (e.g., in the chapter about the formal-abstract perspective, there are questions about the difference between calculus at school and calculus at university, as well as the significance of the concepts of continuity, differentiation, and integration). The key reflecting questions related to the final reflection on perspectives on calculus comprise the last exercise in the book:

> Take your time and reflect on your domain-specific beliefs of calculus and the teaching of calculus. To what extent were aspects of this didactics of calculus new to you? Which aspects do you want to pay special attention to for your future calculus lessons? (authors' translation)

All participants contributed a detailed reflection and were able to differentiate between the perspectives as well as to connect them, referring to the integrated framework of linking domain-specific beliefs with the SDE concept. The following response from a student is prototypical of the responses of the participants:

> Some aspects that were addressed here in the didactics of calculus were, perhaps, known to some extent (such as the different perspectives at school and university), so you had an idea, but it was very enriching to really see how different the approaches were. You might have experienced the consequences yourself. Some aspects such as the toolbox perspective were very familiar from high school ["Oberstufe"], especially those of the "Kurvendiskussion" [explanation: an examination of a function graph by calculating extreme values]. Higher-level constructs such as the SDE or "Grundvorstellungen" were new and offered a good opportunity to reflect for oneself on how one wants to later approach teaching on various topics. It has become important for me to link SDE and to offer the students a good structure that makes it easier to understand complex topics on the basis of areas that have already been covered. (authors' translation)

6 Summary and outlook

The aim of this paper was to present a new approach to subject-matter didactics and to explain it using the context of calculus. For this purpose, the designs of a book and a seminar were presented. The short glimpse into the reflections of some students demonstrated that it is worthwhile to address different perspectives on the concepts, activities, and theorems and to make the differences explicit to encourage a deeper understanding and reflective perspectives on calculus. This can also mean that it might be advantageous to modify standard approaches for teaching calculus. For instance, this opens possibilities to not just strictly follow the systematic structure of calculus. However, the corresponding changes also create new challenges. For example, the systematic structure of the concepts in calculus moves into the background and the students have less awareness of it. The book described in this paper is scheduled for publication in 2023.

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