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Research article

Second graders' multimodal reasoning in playful inquiry-based mathematics activities

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Abstract: This qualitative study aims to provide insights into lower primary students' multimodal reasoning from a sociocultural perspective, examining their use of diverse semiotic means of objectification. The pedagogical approach of playful learning was combined with an inquiry approach to engage groups of seven-year-olds in mathematical activities designed collaboratively with participating teachers. Video recordings were generated and transcribed, focusing on participants' dialogue and actions, and analysed using a reflexive thematic approach. The study shows that the second graders' multimodal reasoning was characterised by the integration of reasoning words and deixis in speech, synchronised with gestures and the use of artefacts during interactions with peers and the teacher. These three semiotic means emerged as the most prominent cultural tools employed by the students to direct attention, establish shared meanings, and actualise arithmetic knowledge when solving the initial problem and extending it through their own elaborations. The teacher's balanced involvement supported the reasoning process by responding to the students' problem expansions and fostering a sense of autonomy in decision-making. Implications are drawn regarding the teacher's role in guided play.

Keywords: elementary mathematics, inquiry, multimodal reasoning, playful learning, semiotic means

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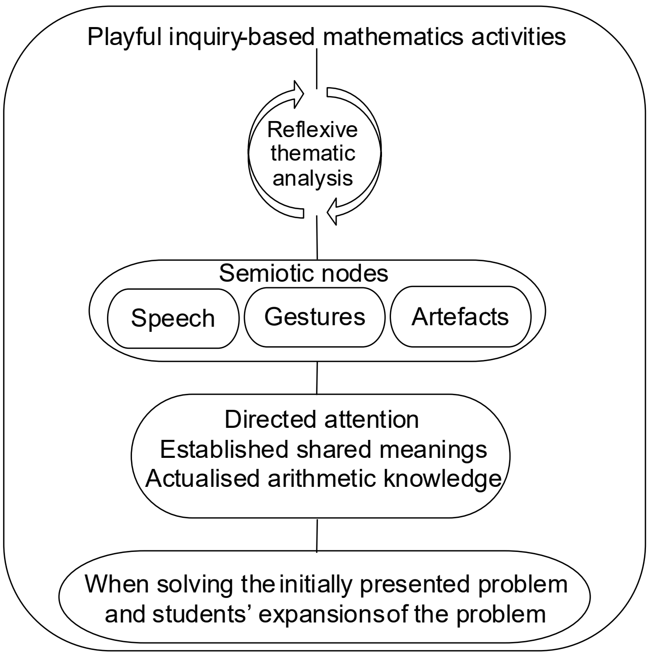

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1 Introduction

In recent decades, researchers have increasingly recognised the importance of communication in joint meaning-making processes (Mercer & Littleton, 2007; Planas & Pimm, 2024; Wells, 2007). There has been a shift in mathematics education towards learning through peer dialogue (Sfard & Kieran, 2001; Webb et al., 2019; Weber et al., 2008; Wester, 2021). Researchers have also highlighted the significance of how thinking is conveyed through nonverbal communication (Alibali & DiRusso, 1999; Bjuland et al., 2008; Radford, 2003; Wells, 2002).

With regard to young students' mathematics learning, previous research has demonstrated that they use a variety of semiotic resources to reason and convey their thinking, including verbal language, gestures, and concrete materials (Johansson et al., 2014; Nergård, 2023; Sumpter & Hedefalk, 2015; Wathne & Carlsen, 2022). Consequently, studies have focused not only on speech but also on other specific verbal and nonverbal resources such as gestures, writings, drawings, symbols, and pictures (e.g., Nergård, 2023; Nordin & Boistrup, 2015; Reynolds & Reeve, 2001; Wathne & Carlsen, 2022). As emphasised by Björklund et al. (2020) in their review of early years literature, there remains a need for further research into young children's embodied ways of learning.

In recent years, there has been growing interest in the study of multimodal communication, particularly in the concept of gesturing (Planas & Pimm, 2024), which has been emphasised as having a significant role in mathematics learning (e.g., Bjuland et al., 2008; Radford, 2003). In my study, gestures are understood as what McNeill (1992) refers to as gesticulations—spontaneous movements of the hands or arms that lack symbolic meaning and occur simultaneously with speech. Gestures play an active and important part in how humans think and learn (McNeill, 1992; Radford, 2009) and thus in promoting collective arguments and reasoning.

Planas and Pimm (2024) advocate for future research to address the challenges of multimodal mathematical communication and to explore how various resources intersect. The present study responds to this call by analysing second graders' multimodal reasoning within the context of two playful inquiry-based mathematics activities presented in Section 3.1.

1.1 Previous research on young students' use of speech with nonverbal resources

Research involving younger students has consistently highlighted the importance of combining speech with hand movements in learning various mathematical concepts. These investigations span domains such as counting (Alibali & DiRusso, 1999), arithmetic (Broaders et al., 2007), fractions (Nordin & Boistrup, 2015), proportions (Abdu et al., 2021), and combinatorics (Wathne & Carlsen, 2022).

Nordin and Boistrup (2015) studied fourth graders' explanations and justifications when reasoning multimodally about fractions. Their findings showed that students employed a variety of resources, such as speech, drawings and gestures, in diverse ways, thereby emphasising the importance of teachers attending to younger students' embodied modes of reasoning and communication.

Abdu et al. (2021) investigated the relationship between third graders' speech and hand movements as they manipulated bars on a tablet to represent proportions. The researchers identified tensions or differences of voices, termed dialogic gaps, which manifested in the students' speech but not in their hand movements. Specifically, these gaps arose when students used the same word with different meanings. For instance, one student used the phrase "the same" to refer to simultaneity, adjusting two bars of different heights with a coordinated, fluent, hand movement. Another student, however, disagreed, interpreting "the same" in terms of bar heights rather than simultaneous movement.

Wathne and Carlsen (2022) argued that Norwegian third graders (aged eight to nine) used multimodal reasoning through speech, inscriptions, and gestures, to support their problem-solving process, express their combinatorial thinking, and mediate intended mathematical meanings. Similarly, Nergård (2023) demonstrated how Norwegian five-year-olds communicated their mathematical arguments in multimodal ways while engaged in play, using language, gestures, body language, and concrete objects.

Given the importance of accounting for students' use of various semiotic means to convey their thinking, researchers have called for future studies to focus on embodied ways of learning (Björklund et al., 2020; Planas & Pimm, 2024). Responding to this call, the aim of the present study is to contribute to this field by providing insights into second graders' multimodal reasoning. Specifically, the students' use of different semiotic means is investigated as they explain their thinking while attempting to solve mathematics problems within the context of playful inquiry-based mathematics activities. The study builds upon the work of Jeannotte and Kieran (2017), who define mathematical reasoning as "a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances" (p. 7). The definition is adopted and integrated with insights from Radford (2009), who emphasises that human thinking occurs "*in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols, and tools" (p. 111).

In the present study, multimodal reasoning is understood as a process that unfolds through the participants' dialogues and actions, where students may use two or more modalities to justify their ideas and demonstrate the validity of their reasoning and associated argumentation. While previous research has often focused on how teachers support students' reasoning and argumentation through whole class discussions (e.g., Kazemi & Stipek, 2001; Nordin & Boistrup, 2018; Tabach et al., 2020), this study instead investigates students' multimodal reasoning within small-group teaching settings, an area that, according to Abdu et al. (2021), remains underexplored.

The study contributes to the existing body of research by addressing the following research question: What characterises the multimodal reasoning of eight small groups of second graders during two playful inquiry-based mathematics activities? More specifically, the study provides insights into the students' multimodal reasoning as they reason about properties of integers and whole number solutions that satisfy given conditions.

2 Theoretical framework

A sociocultural perspective on learning and development is adopted as it acknowledges the central role of social interactions, the ways in which sense-making is established through discourse, and the role of language, culturally constructed tools and artefacts, in learning and development (Säljö, 2001; Vygotsky, 1978, 1986).

2.1 The theory of objectification

Radford's (2013, 2021) theory of objectification is adopted, given its emphasis on how individuals use different semiotic means of objectification not only to convey their thinking but also as integral components of their thinking. Learning is viewed as a process of objectification through which the participants in a teaching-learning activity actualise knowledge as knowing. At the same time, the participants themselves change as they are in a process of becoming. There is a dialectical movement between mathematics and individuals, as both are being produced and transformed historically and culturally through social processes as the individuals engage in sociocultural practices.

Important to the present study is the emphasis the theory places on how we as humans think with and through our senses, and thus, how different semiotic means or resources influence and constitute students' mathematical thinking. Following Radford (2002, 2003, 2021), the students and teachers are engaged in sensuous, material, and embodied meaning-making processes. As they engage with mathematical concepts and ideas in the teaching-learning activity, the students build on their previous knowledge and use their senses so that their knowledge of the involved mathematics and the participants themselves become shaped in historically formed ways. Individuals engage in embodied ways to make sense of the mathematics knowledge. This is referred to as the embodied nature of thinking, as semiotic resources are used by individuals not only to mediate thinking but also as part of their thinking (Radford, 2014, 2021).

The semiotic means are used intentionally by individuals "in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (Radford, 2003, p. 41). When participants use semiotic means to make sense of and make the mathematical object—for example, even and odd numbers—accessible, the semiotic resources are intertwined and cannot be separated, as they jointly express the participants' thinking. Through the use of semiotic resources, students gain insight into mathematical concepts that are shaped by history and culture. At the same time, their senses are transformed, since humans think with and through our senses, including both inner and outer speech, gestures, tactility, and actions with artefacts.

Since sensory experiences are part of how people think, it is, according to Radford (2021), important to detect semiotic nodes in the data material. The concept of a semiotic node aligns with the idea of thinking as a sensuous, material process. It refers to "... segments of classroom activity where a complex coordination of various sensorial and semiotic registers occurs in a process of objectification" (Radford, 2021, p. 122). For example, when students in segments of an activity use various kinds of semiotic resources to bring forward a particular meaning or interpretation of an encountered mathematical concept. Detecting semiotic nodes can offer valuable insight into students' objectification process and multimodal reasoning.

In this study, students' and teachers' engagement in the teaching-learning activity is analysed. One component of the teaching-learning activity relates to the organisation and choice of problems to be discussed. Radford (2021) recommends that problems should be framed within a social context that encourages discussion and reflection with others to support collaborative knowledge production. This approach was adopted in the present study by presenting students with mathematics problems within the context of playful inquiry-based mathematics activities, to be elaborated on in section 2.2.

2.2 A playful inquiry-based approach to the learning of mathematics

The choice of a playful inquiry-based approach is supported by meta-analyses demonstrating that guided play contexts have a greater positive effect on early mathematics skills and related concepts compared with either direct instruction or free play, suggesting that this pedagogical design is especially beneficial for systematic, mathematics-based learning tasks (Skene et al., 2022; Weisberg et al., 2013). Therefore, what is defined in the study as playful inquiry-based mathematics activities (hereafter abbreviated as PIMAs) was designed in collaboration with two participating teachers.

Following Hirsh-Pasek et al. (2008), a playful learning approach involves learning through play, which in the present study refers to the learning of mathematics through play. Zosh et al. (2017) characterise playful learning experiences as joyful, meaningful, actively engaging, iterative, and socially active. These characteristics were incorporated when designing activities based on a playful narrative featuring a character called the "Number King". The intent was to create a context that could capture the students' interest and encourage them to actively engage in the PIMAs.

Various playful learning experiences exist (Pyle & Danniels, 2017; Zosh et al., 2018), such as free play, guided play, and games. Adopting a play-based pedagogy in a continuum that allows for teacher-guided elements is recognised as key to supporting academic learning (Pyle & Danniels, 2017). Therefore, the present study adopts a guided play approach, given the growing evidence of its effectiveness (e.g., see Skene et al., 2022 for a review). Through guided play, students actively engage in the learning process and maintain a sense of control in decision-making, while being supported by the teacher in the pursuit of a learning goal (e.g., Weisberg et al., 2013; Weisberg et al., 2016; Zosh et al., 2018). The designed PIMAs contained learning goals, and, in the role of the Number King, the teachers took part in the activity, dressed in a cape and crown, asking for the students' help to solve different problems.

In addition, a playful learning approach was combined with an inquiry-based practice, building on Jaworski (2005, 2006) and Wells (1999) to enable a dialogue between the participants that promotes argumentation and allows for the use of both verbal and nonverbal semiotic means. The intent was for the students to be motivated to actively engage in the PIMAs and in mathematical inquiry—that is, for the students to contribute ideas, build on each other's utterances, inquire into the mathematics, and communicate their sense-making, which young learners are more likely to do when supported by a teacher in playful activities (e.g., Björklund et al., 2018; Skene et al., 2022; van Oers, 1996). In line with research on problem solving in mathematics education (e.g., Lester & Cai, 2016; Liljedahl et al., 2016; Schoenfeld, 1985), the PIMAs were also designed based on the students' prior knowledge. The intent was to present them with problems that were accessible and, at the same time, sufficiently challenging.

3 Materials and methods

This qualitative research study takes an interpretative stance, aiming to provide insight into the characteristics of second graders' multimodal reasoning in playful inquiry-based mathematics activities. The study extends two previously published studies on students' problem solving and learning opportunities (Flaten, 2025a, 2025b) by analysing a different set of data and addressing a distinct research question. As such, the study is part of a broader research project that draws on Cobb et al.'s (2003) work on design research, but the specifics of the methodology are beyond the scope of this paper.

Two teachers were selected through purposive sampling to ensure that they had an interest in adopting a playful inquiry-based approach to mathematics teaching and learning. Purposive sampling was chosen as research shows that teachers' beliefs about teaching may hinder the successful implementation of student-centred approaches (Bubikova-Moan et al., 2019; Fry et al., 2025)—for example, if the teacher has previously experienced success with teacher-centred approaches and doubts the effectiveness of student-centred approaches in preparing students academically.

Eight groups, comprising a total of 35–38 students, participated in the study. One group at a time engaged in the classroom activities together with their teacher until they solved the presented problem, which took between 10 and 18 minutes per group. Each group participated in both PIMAs twice, resulting in a total of four sessions per group. The intended group size was four students, although this occasionally varied due to fluctuations in attendance and student absences. All participants were informed about the purpose of the study and their right to withdraw without any repercussions. Informed consent was obtained from the two teachers and the 38 second graders, including from the students' legal guardians. The participants were assigned pseudonyms, and the recordings were stored securely.

3.1 The design of the playful inquiry-based mathematics activities

The designed PIMAs contained learning goals, and, in the role of the Number King, the teachers took part in the activity dressed in a cape and crown, asking for the students' help to solve different problems. A pilot study was conducted, which informed the design and refinement of the two PIMAs, labelled PIMA1 and PIMA2, that form the basis of the study's analysis and are presented next.

3.1.1 The combination of coins problem

PIMA1 was implemented twice over a three-week period. The students were given a doll to represent princesses and princes of the Number King. They were to figure out the combination of coins locked inside a treasure chest. Through the narrative, the Number King, i.e., the teacher, revealed the sum of the coins, e.g., equal to 37 Norwegian kroner (NOK). When appropriate, three other criteria were also revealed through the narrative. Locked inside the chest were (1) no coins of the value 20 NOK, (2) coins of the value one NOK, and (3) a total of eight coins. To help the Number King figure out the combination of coins, the students exchanged coins in attempts to adapt and adjust their solution to fit the given criteria.

3.1.2 The stolen coins problem

Due to a school holiday, PIMA2 was implemented twice over a seven-week period. In a letter from the Police in the Number King's Kingdom, the students were informed about the mission. They were to interrogate the thief and figure out how many coins, between 1 and 100, he had stolen from the Number King. The thief refused to talk to anyone other than the students. Also, he would only answer ten yes-or-no questions without answering the same question more than once. A free-of-charge text-to-speech app connected to a speaker was controlled by the researcher to allow the students to interrogate the thief. The number of coins varied between groups without the teacher knowing the answer. The students put on detective hats and used paper plates to eliminate numbers on a hundred chart, i.e., a 10-by-10 grid containing numbers from 1 to 100.

3.2 Data generation and analysis

To capture the students' use of various semiotic resources and the context in which they occurred, classroom observations were conducted and video recorded, accompanied by taking field notes. In line with the dialogic approach of Linell (1998), the dialogues were jointly constructed as a result of the participants' collaborative meaning-making in the situated context. Therefore, the utterances must be viewed consecutively and in connection with each other and with the context. A total of nine hours of video recordings were generated. For examples provided in the results section, the recordings from implementing the two PIMAs were transcribed line by line, with a consecutive numbering of each line. To provide detailed and nuanced accounts of the students' multimodal reasoning, actions taking place were included and written in parentheses between the utterances according to when the actions occurred. Emphasised words were underlined, and relevant comments were included to increase readability.

Following data generation, a detailed discursive analysis was conducted. The analytical process resembles a data-driven thematic analysis, drawing on the work of Braun and Clarke (2006) and further elaborated by Braun et al. (2019). The starting point was the empirical data material from implementing both PIMAs, which was analysed without applying a predetermined set of codes, although I acknowledge that I was informed by my preliminary understanding of the literature. In line with the reflexive and recursive processes of thematic analysis (Braun et al., 2019), the process of analysing the data involved going back and forth between various phases of coding, categorising, identifying themes, and revising existing codes and categories. Through the process, similar codes emerged in the data from PIMA1 and PIMA2, which were scrutinised further to identify characteristics of the students' multimodal reasoning across the data from the two PIMAs.

For example, some of the emerging preliminary codes that recurred were 'finger pointing', 'hand pointing', and 'foot pointing', as students pointed with an index finger, with a foot, or by laying their hand down at a symbol or object. These are examples of gestures McNeill (1992) classifies as deictic gestures. Following McNeill (2005), gestures may serve more than one meaning. However, the second graders' three pointing gestures all appeared to serve the purpose of pointing and were combined as the code 'pointing' and categorised as a 'gesture'. The decision was based on their consistent communicative function across different instances, as each gesture was used by the students to indicate or draw attention to specific objects or representations relevant to the mathematical problem. As such, following Linell (1998) and a data-driven thematic analysis (Braun et al., 2019), gestures were analysed in relation to their communicative intent and contextual use, which aligns with the study's aim. 'Sliding' is another example of a code in the category 'gestures', e.g., used when students were observed sliding coins when calculating the total sum or making a sliding gesture under the digits of a number or through a column of the hundred chart. As such, both 'pointing' and 'sliding' gestures could occur together with the use of 'artefacts', as another semiotic means of objectification.

Also, the code 'pointing' occurred together with the participants' use of words like 'there', 'here', 'these' and 'those'. These are examples of words that indicate location and which Radford (2002) terms 'deixis', i.e., words related to the action of pointing out or indicating something. The students also used 'reasoning words' or 'discourse markers' like 'if', 'so', 'because', 'therefore', 'but', or they stated, "I think…", "I agree…" or "I disagree…". Hence, included in the category 'speech' are codes related to the students' wording, such as the use of deixis and reasoning words.

The above mentioned codes, that eventually were categorised as the semiotic means of 'speech', 'gestures' and 'artefacts', occurred together and were prevalent in the data material across both PIMAs. After revisiting Radford's (2021) theory, and particularly the concept of semiotic nodes, the data material was further scrutinised to analyse the semiotic nodes consisting of the three prevalent semiotic means, as they characterised the students' multimodal reasoning across the two PIMAs.

4 Results

The excerpts presented in the results section are chosen, not because they necessarily illustrate the playful aspects of the students' involvement in the PIMAs, but because they are representative examples of broader patterns in the data. The presented excerpts demonstrate the students' multimodal reasoning and its characteristics across groups of students from both classes during the implementation of both PIMAs. Each subsection begins with a heading that signals the thematic focus and direction of the results and its relevance to the broader study.

4.1 Synchronise the use of speech, gestures, and artefacts to argue when comparing different solution suggestions to the presented problem

Excerpt 1 is from the first activity, PIMA1, where students exchanged coins in their attempts to adjust for the given criteria and figure out the combination of coins locked inside the treasure chest. The excerpt is chosen as it provides a representative example of the students' multimodal reasoning through the use of speech, gestures, and artefacts to compare the group's different suggestions for a solution. Prior to the excerpt, the chest with coins had been introduced to the narrative, but the group had not been given any of the criteria of the problem yet, not even the sum of coins locked inside the chest. At this point, they are discussing the differences between the group's various combinations of coins equal to 30 NOK. Sophie has one ten, two fives, and ten ones. Olivia has two tens and ten ones. Noah has two tens, one five, and five ones.

Excerpt 1. Students synchronise the use of speech, gestures, and artefacts to argue as they compare different solution suggestions to the problem

Line number	Speaker, pseudonyms for the students	Utterance (with actions and comments)
84	Sophie:	We are satisfied (jumps up and down with her doll next to her coins)
85	Teacher:	Okay (raises a thumb to the chin, looks at the coins in front of Olivia and Sophie while raising the eyebrows). Which one should I choose then?
86	Olivia:	But we cannot know that
87	Teacher:	Because if we are wrong… (Does not finish)
88	Sophie:	But I do not really think it is mine because she (points at Olivia) has the least money
89	Teacher:	She has the <u>least</u> money? (Questioning intonation. Furrows the forehead)

90	Noah:	I think it is like this, two tens (slides the tens in front of himself), a five (slides the five next to the tens) and five ones (slides the ones next to the tens and five, after which he uses both hands to slide all the coins further in on the table, away from himself)
91	Sophie:	Yes (looks at the teacher), but these (holds up to fives) are exchanged to a ten because there are only one of them (points to the two tens in front of Olivia). I have one krone (NOK) more than she has (looks and points at Olivia)

Sophie expresses satisfaction (line 84) after the group has provided different combinations of coins equal to 30. It should be noted that the students were familiar with a different activity where the goal was to provide as many combinations of a number as possible. The group's three suggestions of a combination equal to 30 may explain Sophie's satisfaction. The teacher responds by asking which combination they should choose (line 85). Olivia is correct in her claim that they cannot know that (line 86). At this point in the activity, it could be any combination of coins inside the chest because the group has not yet been given all the criteria of the problem yet. The teacher does not finish his sentence (line 87), interpreted as a rhetorical pause. Indirectly, he problematises the information that they have previously been given: the group should get it right because they can earn the money. Sophie responds by using a reasoning word while pointing at Olivia, i.e., she coordinates speech with a gesture. She argues that it cannot be her combination because Olivia has the least money (line 88). The teacher questions Sophie's claim (line 89) by repeating it in a questioning intonation, while emphasising "least" and furrowing his forehead. Noah disagrees as he argues for his suggestion (line 90). Noah verbalises his combination of coins, coordinated by the gesture of sliding the coins in front of himself—first the tens, then the fives, then the ones, and lastly, he slides all the coins further into the table and away from himself. Sophie then responds to the teacher's question. She argues verbally accompanied by picking up coins and holding them up, while saying, "Yes, but these are exchanged to a ten…" (line 91). The difference between Sophie and Olivia's suggestions is precisely that Sophie has two fives compared with one of the two tens of Olivia. Sophie continues backing her argument using the reasoning word "because", as she adds: "… because there are only one of them (points to the two tens in front of Olivia), I have one krone (NOK) more than she has (looks and points at Olivia)" (line 91).

Despite Sophie confusing the number of coins with the Norwegian word "kroner", that is, the sum in Norwegian kroner—she is right in that one ten compared with two fives is what differs between the two girls' suggestions. Compared with the observations of students pointing at coins in PIMA1 or pointing at number symbols in PIMA2, Sophie synchronises her speech with the gesture of picking up and holding the coin up in the air in between herself and the teacher.

As such, Excerpt 1 exemplifies how the students reasoned in a multimodal manner by synchronising the use of speech, gestures, and artefacts to argue for and compare their different solution suggestions to the initially presented problem. As illustrated within the semiotic node of Excerpt 1, students did so in different ways. They synchronised speech with the gestures of pointing at, sliding, and holding artefacts, i.e., coins, up in the air to direct the participants' attention, create shared common ground, and convey their thinking.

4.2 Students build on peers' utterance and reason in a multimodal manner to strategically analyse whole numbers

Excerpt 2 is from PIMA2, where the students are trying to figure out the number of coins the thief stole. The excerpt is chosen as it provides a representative example of the students' multimodal reasoning when they, with limited involvement from the teacher, were observed strategically analysing numerical similarities and differences. In PIMA2, they did so to reach a conclusion regarding which question they should *not ask* the thief. The process of deciding which question *to ask* the thief, sometimes involved discussions about what question the group should *not ask*, and *why*. The group was standing around the hundred charts, looking for similarities and differences between the even numbers from 46 to 100 that remain on the chart.

Excerpt 2. Students build on peers' utterance and reason multimodally to strategically analyse whole numbers

Line number	Speaker, pseudonyms for the students	Utterance (with actions and comments)
5400	Lily: [...]	I do not want to ask if it is (a) three-digit number. Because it is not ... (Unfinished, walks over to the chart, points with foot on 100). Because it is only one.
5403	Emma:	Not that it is a two-digit number, because all are two digits.
5404	Lily:	And it is only one three-digit (points again on 100 with her foot), so that is not smart.
5405	Teacher:	No, that is true.
5406	Lily:	Because there are most like these (from where she is standing below the chart, she points with her index finger in the direction of 78 and 66, spatially in the middle of the chart).
5407	Teacher:	Mm (everyone looks at the chart). But is there anything else we could ask, which can be the same in several numbers here?

Lily contributes to the group's collective reasoning process by arguing that she does not want to ask if it is a three-digit number (line 5400). She backs up her argument by using a reasoning word saying, "Because it is not ...", before she points at 100 on the hundred chart, and continues to back up her argument with, "Because it is only one (three-digit number)", while using a pointing gesture. As such, Lily compares the one three-digit number to the other numbers left on the chart. Emma's response (line 5403) can be seen as a continuation of the collective reasoning process, as Emma builds on the idea of comparison as well as on Lily's utterance. Emma directs the group's attention towards two-digit numbers, pointing out that they should not ask about two-digit numbers either,

"… because all are two digits" (line 5403). Lily is building on Emma's claim about two-digit numbers which is signalled by the word 'and' as she responds, "And it is only one three digits, so that is not smart" (line 5404), accompanied by a pointing gesture at 100. The teacher supports the previous arguments (line 5405). Lily concludes by emphasising the previous comparison of the remaining numbers, "Because there are most like these" (line 5406), while gesturing towards two-digit numbers in the middle of the chart. With a supportive "Mm" and a prompt from the teacher (line 5407), the group continues discussing other similarities and differences between the remaining numbers.

Excerpt 2 provides a representative example of how the second graders participated in a mathematical discourse and collectively reasoned, as they built on each other's utterances, with limited involvement from the teacher. The students were observed identifying similarities and differences between the remaining numbers and comparing these through the use of speech with reasoning words, accompanied by pointing gestures towards numbers on the chart, i.e., the use of an artefact. During the implementation of both PIMAs, students were observed building on each other's utterances as they conveyed their thinking using speech with reasoning words or signalling markers, i.e., 'because' or 'so…', together with pointing gestures towards numbers on the chart.

As illustrated within the semiotic node of Excerpt 2, the students used the three intersecting semiotic means of speech, gestures and artefact to exchange views and collectively argue in a multimodal manner. They did so to reach the conclusion that they should not ask about three- or two-digit numbers, because there are mostly two-digit numbers left and only one three-digit number. This is an important step in the problem-solving process because it brings the group one step closer to figuring out what question they should ask, and why. As Excerpt 2 illustrates, it seemed as if the students did not want to ask a question based on a similarity or difference that most or only a few (i.e., one) numbers have in common. Such observations were made in other groups as well, where students built their arguments based on asking questions that strategically would lead to the exclusion of approximately half or "many", i.e., not all or just a few of the remaining numbers, regardless of the thief's confirmative or negative response to their question.

4.3 Students reasoned in a multimodal manner to solve their own expansion of the presented problem

Excerpt 3 includes a sequence from the transcripts after one group has interrogated the thief and solved the presented problem in PIMA2. The group has figured out that the thief stole 24 coins of the value 10 NOK. One student, Liam, is observed counting on his fingers before he takes the initiative to expand the presented problem. The excerpt provides an example of students' multimodal reasoning as they take initiative to expand the initially presented problem. The interplay between Liam's speech and gestures of pointing at number symbols on the chart is exemplified as he reasons in a multimodal manner to convey how he calculated the total sum of stolen coins as an expansion of the presented problem.

Excerpt 3. Students reasoned in a multimodal manner to solve their own expansion of the presented problem

Line number	Speaker, pseudonyms for the students	Utterance (with actions and comments)
6393	Liam:	Oh, that was a lot of money (lowered voice).
	[…]	(The others talk amongst themselves)
6396	Liam:	It was 240 money! (Raised voice)
6397	Teacher:	240, what? (Looks at Liam)
6398	Liam:	Money, he took, or… (Turns, looks at the chart, unfinished sentence)
6399	Teacher:	240 (Norwegian) kroner? (Questioning intonation)
6400	Liam:	Yes? (Turns and looks at T)
6401	Teacher:	How did you know that? (Looks at Liam)
6402	Liam:	Because (runs over to the hundred chart). This here (points with index finger at 2 in 24), that makes it so that it becomes 200 and that (points with index finger at 4 in 24) makes it so that it becomes 40.
6403	Teacher:	Ah… Because it was ten coins (which the thief stole)?
6404	Liam:	Mm.
6405	Teacher:	Oh, you… How clever.

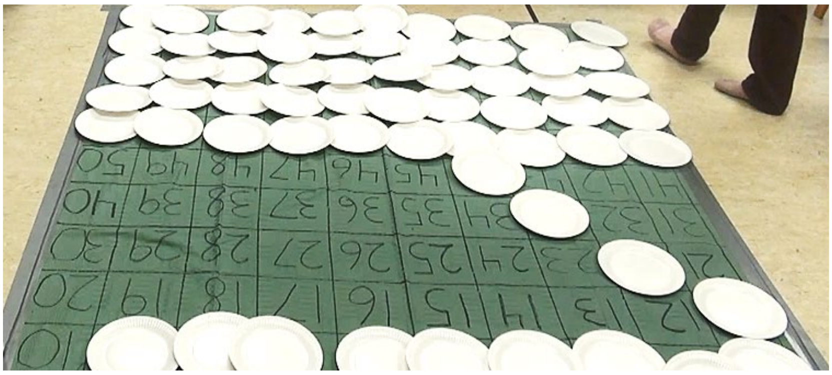
On his own initiative, Liam expands the problem of PIMA2. First, in a lowered voice (line 6393), before he raises his voice to get the others' attention and expresses the total amount of the 24 stolen ten coins (line 6396). The teacher directs attention towards Liam's claim by asking a "how"-question (line 6401). Liam reasons in a multimodal manner and conveys his thinking using speech with reasoning words, synchronised by the gesture of pointing at signs on the chart, i.e., to the tens and units of a number symbol on the chart (line 6402). In doing so, Liam creates a shared common ground as he supports his claim (from line 6393) and uses the property of distributivity to convey the general structure behind the calculation to the others, that is, $10 \cdot 24 = 10 \cdot (20 + 4) = 10 \cdot 20 + 4 \cdot 10 = 200 + 40 = 240$.

As such, the semiotic node of Excerpt 3 provides an example of a student who reasoned in a multimodal manner using speech, pointing gesture, and artefacts as key semiotic resources, to justify a claim with regard to the students' own expansion of the presented problem.

4.4 Students in the process of becoming aware of encoded forms of thinking and doing

The empirical data includes instances where the participants struggle to establish shared meanings. Excerpt 4 serves as a representative example of this issue. Figure 1 presents the remaining numbers on the chart, which the teacher, prior to Excerpt 4, had directed the students' attention towards, to discuss what question they should ask next. The excerpt is chosen as it provides insights into the students' process of becoming aware of encoded forms of thinking and doing.

Figure 1. Excluded and remaining numbers prior to Excerpt 4



Excerpt 4. Students in the process of becoming aware of encoded forms of thinking and doing

Line number	Speaker, pseudonyms for the students	Utterance (with actions and comments)
3970	Mia:	Oh, is it ten or twenty or thirty or... fifty? (Pointing gesture towards 10, 20, 30, 40, 50)
3971	Peter:	Then it has to be fifty
3972	Mia:	We could ask about that? (Questioning intonation)
3973	Teacher:	But can, he can only answer yes or no? (Questioning intonation)
3974	Mia:	Yes, if he, we can, if he says yes, or no, then we can take away all (steps at the tenth coloumn at 10-50, turns, looks at the other numbers that remain on the chart), but if he says yes, then we can, if he says no, then we take away all of those (waves with both hands over 10-50) and if he says yes, then we take away all the others (waves with both hands above the other remaining numbers)
3975	Teacher:	Okay? Which one should we take then?
3976	Mia	Uhm
3977	Teacher:	If it is something with...? (Questioning intonation)
3978	Mia:	Is it something with ten, or twenty, or thirty or forty or fifty? (Looks at each number synchronised with speech)
3979	Teacher:	Yes, but it is that regardless, isn't it?
3980	Mia:	I mean, is it ten, or twenty, or thirty, or forty, or fifty? (Pointing gesture above the respective numbers on the chart)
3981	Teacher:	He can only answer yes or no? (Questioning intonation)
3982	Mia:	Yes, then he says... (Puts arms out in front of her, looks at column with 10, 20, 30, 40 and 50 remaining)
3983	Teacher:	Okay, try
3984	Phil: [...]	Okay! (The group walks towards the speaker, asks the questions, and the thief answers no)

3989	Phil:	Now you said it, and it is not… (Looks at Mia, smashes his hand in the table and drops both arms down on his thighs while sighing)
3990	Mia:	Yes? Now, then we can put away all those (points at 10, 20, 30, 40 and 50), except those (turns, waves with a finger over the remaining numbers). Then we can take away all of those (points at 10, 20, 30, 40 and 50).
3991	Teacher:	Oh, yes, you thought, all that had zero in the unit's place? (Questioning intonation)
3992	Mia:	Yes! (Raised voice)
3993	Teacher:	Ye̲s… (Long e). Then you must cover all of those then (students grab plates).
3994	Mia: […]	I am clever sometimes (smiles, students exclude 10, 20, 30, 40 and 50 by putting down plates)
4000	Mia:	It is good that you have such a clever detective (the teacher laughs and confirms)

One student, Mia, suggests that the group ask if the number of stolen coins is 10, 20, 30, 40, or 50 (line 3970). The group discusses the question together with the Number King (i.e., the teacher) who problematises the fact that the thief can only answer yes or no (line 3973). Still, Mia insists on asking the question.

Building on key concepts in the theory of objectification (Radford, 2002, 2013, 2021), the excerpt provides a glimpse into the students' process of becoming aware of encoded forms of thinking and doing. Within the semiotic node of Excerpt 4, Mia's use of speech with the reasoning word "if" and deixis of "those" and "these" is synchronised with pointing gestures at or towards number symbols on the chart. Following Radford (2003), even though the different semiotic means of objectification are culturally influenced, the meaning may not be clear to students. Engaged in the teaching-learning activity, Mia actualises mathematical knowing through the use of speech with deixis, synchronised with pointing gestures towards number symbols on the chart. She argues that if the thief answers no, they can get rid of 10, 20, 30, 40, 50, and if he answers yes, they can get rid of all the other numbers left (line 3974). She considers and builds her justification based on all possibilities, that is, the possibilities of receiving either a confirmative or negative response from the thief. Mia's multimodal reasoning indicates that what she attempts to communicate is a suggestion to ask about "all of those" (that is, 10, 20, 30, 40, 50) compared with "all the others", i.e., all the other remaining numbers on the chart. Following Radford (2002), Mia uses speech with deixis and pointing gestures to draw attention to specific numbers that have a similarity, namely that they all contain zero in the unit's place (as opposed to "all the others").

However, Mia's speech is imprecise when she tries to convey the suggested question (line 3970, 3978, and 3980). Mia's gestures synchronised with speech in line 3974, reveal more insight into her sense-making regarding what the numbers 10, 20, 30, 40, and 50 have in common, compared with when she objectified her knowing and articulated the question solely using speech in line 3970, 3978, and 3980. As Mia synchronises speech with pointing gestures in line 3974, she helps fill the verbal gap as she adds information to the deixis "those" and "the others" in her attempt to establish shared meaning. Whereas in lines 3970 and 3980, she suggests asking if the number is 10, 20, 30, 40 or 50, i.e., specific numbers. In line 3978, her speech communicates a desire to ask if the number of stolen coins is "… something with 10, 20, 30, 40 or 50". The latter goes for all the remaining numbers (see Figure 1), which might explain the teacher's response in line 3979. Mia and the teacher's perspectives seem not to be aligned when it comes to the similarities and differences between the remaining numbers that Mia is trying to convey.

What Mia attempts to convey in a multimodal manner is related to the conceptual content of the mathematical term whole numbers, more specifically whole tens. The semiotic node consisted of a complex coordination of speech with deixis and reasoning words, accompanied by the use of gestures and artefacts. Following Radford (2021), identifying semiotic nodes provides insights that have implications for students' consciousness, as learning is about the creation of consciousness, and about becoming conscious, into which semiotic nodes provide insights.

After attempts to clarify and become attuned to each other's perspectives (lines 3975–3982), the necessary level of shared meanings is not established. The excerpt illustrates that Mia's effort to coordinate and link speech with the use of gestures and artefacts is not enough to establish shared meaning, as it requires that the listeners perceive and make sense of the semiotic means. The teacher proceeds by agreeing that they should try to ask the suggested question (line 3983).

By agreeing, the teacher lets the students maintain a sense of control in decision making, in line with principles of playful learning that are evident in guided play (Weisberg et al., 2013; Weisberg et al., 2016; Zosh et al., 2018). After observing Mia's actions of excluding numbers, the teacher makes sense of the meaning that Mia has tried to convey. The teacher responds as if she is having an aha-moment (line 3991), which supports the previous interpretation of a lack of shared meaning and enriches the learning experience. Mia adds to the establishment of shared meaning by articulating and pointing at numbers to be excluded (line 3990), after which the teacher introduces the semiotic means of a precise mathematical language with regard to the mathematical concept of whole numbers (line 3991). Mia confirms, smiles, and calls herself a clever detective (line 3992, 3994 and 4000), which is interpreted as an expression of satisfaction and confirmation of an establishment of shared meanings, through which a glimpse into the students' encounter with cultural-historical ways of thinking and acting is provided.

5 Discussion and conclusion

The study pursues the research question: What characterises the multimodal reasoning of eight small groups of second graders during two playful inquiry-based mathematics activities? By focusing on semiotic nodes, the analysis of the second graders' usage and linkage of semiotic means of objectification provided insight into their knowledge production, i.e., their multimodal reasoning, when engaged in problem solving within a playful inquiry-based context.

The observed and analysed situations show a specific pattern of multimodal reasoning: the students' reasoning was primarily characterised by their use of the intersecting semiotic means of speech (featuring reasoning words like 'because' or 'so' and deixis such as 'these' or 'those') synchronised with pointing and sliding gestures and the use of artefacts (such as coins and the hundred chart). These three semiotic means were the most prominent cultural tools used by the students to direct attention, establish shared meanings, and support their reasoning in dialogues with peers and the teacher. This coordination was not merely expressive; it was integral to the reasoning process, fulfilling an objectifying role.

A synthesis of the key results reveals specific patterns in how these semiotic resources operated integrally during the students' mathematical reasoning, to establish shared foci of attention and meaning. The students used reasoning words and deixis in speech and gestures to explicitly present claims, argue, and compare alternative solution suggestions (cf. Excerpt 2), illustrating that the interplay between gesture and discourse mediated mathematical reasoning among the students. The coordination of modalities fulfilled an essential compensatory and justificatory role. For instance, when students faced conceptual difficulties or lacked a precise verbal language to express complex ideas—a common occurrence for young learners (e.g., Bjuland et al., 2008)—their coordinated gestures and use of artefacts helped to bridge this verbal gap and establish shared meanings. This resonates with the broader understanding that gestures can reveal implicit knowledge, making the student more receptive to learning and promoting conceptual shifts (e.g., Broaders et al., 2007).

Furthermore, the study demonstrates how students initiated higher-level reasoning, such as directing and expanding the initial problem (cf. Excerpt 3), through the combined use of these three semiotic means, supporting their indispensable role in supporting cognitive and communicative acts. Previous research has demonstrated the significance of multimodal engagement in mathematics learning, often focusing on verbal language, gestures, and concrete materials used by younger students (e.g., Johansson et al., 2014; Nergård, 2023; Sumpter & Hedefalk, 2015; Wathne & Carlsen, 2022). The present study supports and extends these findings by providing empirical examples that illustrate how and why these resources intersect as fundamental components of reasoning, in line with the emphasis on embodied ways of learning (Björklund et al., 2020; Planas & Pimm, 2024). By showing how these modes compensate for linguistic difficulties in small-group settings, the study strengthens the empirical foundation for adopting multimodal perspectives in early years mathematics research and teaching.

Specifically, this research situates itself within the broader landscape by deepening the understanding of the coordination of semiotic registers within semiotic nodes (Radford, 2021), moving beyond mere documentation of resource use to interpretation of the objectification process itself. The study also contributes to existing research by exemplifying how the students and teachers, not only worked 'in groups', but worked 'as a group', that is, they worked 'with' each other and built on each other's ideas and utterances in line with multimodal reasoning and Radford's (2021) concept of joint teaching-learning activity. Also, the participants did so as they addressed an area that according to Abdu et al. (2021) remains relatively underexplored: multimodal reasoning within small-group teaching settings.

Moreover, the distinct roles of different gestural movements support and extend the nuanced insights provided by research on older students (Wathne & Carlsen, 2022). While this study's second graders used pointing gestures alongside speech and artefacts, primarily to establish shared foci of attention on the problem or inscriptions, the observations also reveal justificatory moves akin to the sliding gestures noted in Grade 3 students in the study by Wathne and Carlsen (2022). In combinatorial problem-solving contexts, the sliding gesture helped mediate mathematical meaning and systematically justify reasoning when students' inscriptions were incomplete (Wathne & Carlsen, 2022). As Broaders et al. (2007) noted in arithmetic context, forcing children to gesture encourages the expression of new problem-solving strategies, even if only implicitly.

Furthermore, the data confirmed the importance of teachers paying attention to a multimodal perspective, particularly where students, despite struggling with precise verbal language, used coordinated gestures and artefacts to help fill the verbal gap and establish shared meaning, such as the effort to communicate the concept of whole tens (cf. Excerpt 4). The study provides insights into how the students engaged together with peers and the teacher in the teaching-learning activity to actualise mathematical knowledge and became aware of encoded ways of thinking and doing. Thus, the study's results support other researchers' emphasis on the importance of paying attention to a multimodal perspective in research and teaching (e.g., Bjuland et al., 2008; Radford, 2003), including with the youngest in school (e.g., Nordin & Boistrup, 2015; Wathne & Carlsen, 2022). The three intersecting semiotic means fulfilled an objectifying role, strengthening and making the students' thinking apparent, and were, in themselves, used to convey the students' sense-making, thereby achieving a stable form of awareness. Thus, the study reinforces that capturing the entirety of children's mathematical thinking requires attending to this broad range of communicational modes (Björklund et al., 2020; Nordin & Boistrup, 2018).

Moreover, it aligns with Radford's (2002, 2021) theory, which posits that semiotic resources are used intentionally by individuals in social meaning-making processes. As they engaged in the process of solving mathematical problems, the students drew upon their prior knowledge and multimodal resources and made connections between different semiotic means. The presented excerpts illustrate how the students contributed with their ideas and arguments, were explicit about their claims, explained, justified and built upon peers' utterances as they engaged in discussions of deciding whether their ideas were appropriate and collectively reasoned. For instance, the students used multimodal coordination of speech, gestures, and artefacts, to argue for and compare different solution suggestions (cf. Excerpt 1), and to strategically analyse whole numbers by identifying similarities and differences (cf. Excerpt 2). The results resonate with the framework of multimodal dialogue, which conceptualises dialogue using a multimodal lens, recognising that differences in multimodal voices drive communication and learning. The excerpts demonstrate how the students collectively reasoned by building on each other's ideas, which is a process essential for co-constructing mathematical knowledge and consistent with dialogic approaches (Linell, 1998).

Also, it is worth considering that the second graders had to interpret and make use of the semiotic means of objectification in a short time. Within that time, the students had to use appropriate semiotic means to convey their thinking and interpret the semiotic means used by others. Nonetheless, despite the time constraints and the students' age, the use of nonverbal resources was essential for the students to convey their thinking and establish the necessary degree of shared meaning in dialogue. The study illustrates that the students' multimodal reasoning was a result of their effort to make sense of, apply, and combine, mathematical concepts and procedures during the process of collaboratively solving the problems.

In addition, the students' multimodal reasoning was characterised by the use of the three semiotic resources, not only to drive the reasoning forward to solve the initially presented problem, but also as they directed and expanded the problem of the PIMAs by posing and solving their own problems. The study, therefore, also illustrates an important observation that was made in the implementation of both PIMAs, that is, the teachers allowed the students to try out their ideas and direct the activity. This is consistent with principles of playful learning and guided play, which emphasise the importance of teachers providing students with freedom and flexibility in decision making and to balance their own and the students' participation (van Oers, 1996, 2010; Weisberg et al., 2013; Weisberg et al., 2016; Zosh et al., 2018). The study therefore contributes with insights regarding mathematical communication within specific pedagogical contexts, particularly small-group work using a playful inquiry-based approach. By focusing on eight small groups, the research responds to the need for deeper investigation into multimodal reasoning in peer-to-peer contexts (Abdu et al., 2021). The teacher's balanced involvement, responding to student expansions and allowing decision-making control, contributed significantly to supporting the complex multimodal reasoning process observed. Thus, the limited yet supportive involvement of the teacher, who guided the students toward encoded ways of thinking and doing and prompted the clarification of meaning (cf. Excerpt 4), highlights the role of the adult as a crucial dialogue collaborator. This is consistent with findings showing that guidance from adults is more likely to help young learners gain more extensive and explicit investigated mathematical ideas, particularly when teachers adopt responsive actions like asking open questions or extending mathematical content within the play frame (e.g., Björklund et al., 2018; van Oers, 1996).

Therefore, the present study has important implications for classroom practice as it provides examples of the importance of allowing the students to maintain control and direct the activity. To provide students this agency can result in them actualising mathematical knowledge and in the establishment of shared meaning, as they solve the presented problem or their own expansions of the initially presented problem. Given the call made by Planas and Pimm (2024) for more research on challenges of multimodal communication, the study provides implications with regard to the potential of a childcentred approach in situations where students and teachers experience difficulties in their multimodal communication. Teachers need to focus on students' use of different semiotic means of objectification and let them direct the mathematical activity, otherwise might important indications of their learning processes go unnoticed.

While the study offers valuable insights into students' multimodal reasoning during collaborative group work, its small sample size and focus on a specific age group limit the generalisability of the results to a broader educational context. The study's qualitative design prioritises depth. The results are contextually rich and not intended to represent all forms of mathematics learning or group interaction. These limitations do not undermine the validity of the results but rather highlight the importance of situating them within their specific context.

Based on the above discussion, I hold that the PIMAs together with the teachers' implementation of these, supported the students' learning of mathematics as a socially mediated process that encouraged the second graders to put their semiotic resources into use in solving problems at an early stage of their schooling and mathematics learning. Given the insights provided, I argue that a playful inquiry-based approach to small-group teaching offers a valuable way of engaging students in both multimodal reasoning and mathematical content in lower primary school. Agreeing with other researchers (Björklund et al., 2020; Planas & Pimm, 2024), it is an approach that deserves further attention from both researchers and educators. Future research could explore multimodal reasoning across diverse age groups, educational settings, and mathematical problems to further examine its role in collaborative learning settings.

Research ethics

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Informed consent statement

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Data availability statement

Data are unavailable due to privacy or ethical restrictions.

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Conflicts of Interest

The author declares no conflicts of interest.

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