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Research article

Prospective primary school mathematics teachers' professional knowledge of the 'House of Quadrilaterals'

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Abstract: Mathematics teachers ought to possess solid knowledge of different school mathematics topics. Though classification of quadrilaterals ('House of Quadrilaterals') is an important and integral part of geometry curricula worldwide, research reports on learners' difficulties understanding this particular geometric topic. The goal of this paper was to examine what professional knowledge on concept formation, using the example of the 'House of Quadrilaterals', prospective primary school mathematics teachers exhibited after attending a two-semester geometry course. For this purpose, an exploratory mix-methods study with 95 prospective primary school mathematics teachers was conducted. Capturing different facets of their professional knowledge of the aforementioned topic was based on analyzing a single task with three sub-tasks on concept formation. Descriptive statistics revealed satisfactory achievement with the majority of the prospective primary school mathematics teachers achieving more than half of the points. The results of the qualitative content analysis revealed an in-depth insight into a wide range of competence levels in all facets of professional knowledge. In the context of adopting one student's conception of a square and (general) quadrilateral, identifying possible causes of underlying student's misconceptions, and describing theory-driven suggestions for their prevention, deficits in prospective primary school mathematics teachers' content and pedagogical content knowledge were evident. Especially, establishing links between specific sub-tasks and the subject-specific facets of professional knowledge proved to be challenging. Consequently, implications for mathematics teacher education pertaining to (re)design of courses that would support the development of prospective teachers' professional knowledge, are provided. Furthermore, alternative study designs are discussed, which would enable a more in-depth assessment of prospective primary school mathematics teachers' professional knowledge.

Keywords: hierarchical classification of quadrilaterals, mathematics teacher education, primary education, professional knowledge, prospective mathematics teachers

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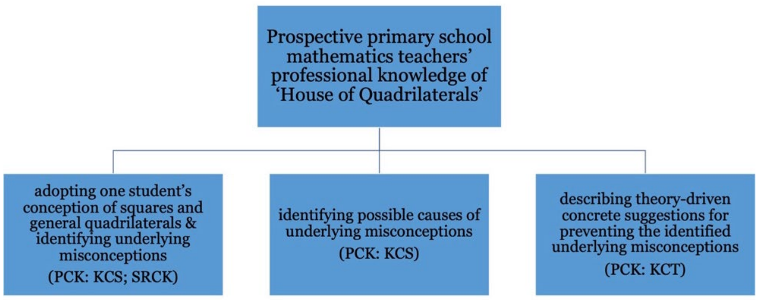

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1 Introduction

> "Actually, I thought I was ... relaxed and calm because the 'House of Quadrilaterals', with just the arrows, can't be that difficult, but you have to be really careful. It always worked out quite well in my homework, but (laughs) not so much in the exam."

This statement by a prospective primary school mathematics teacher is indicative of the experiences of teacher trainees who have taken the final exam in the 'Geometry and Teaching Geometry' course over the last few years. Numerous empirical studies (de Villiers, 1994; Fujita, 2008; Fujita & Jones, 2007; Heinze, 2002; Monaghan, 2000; Pickreign, 2007) confirm that not only students at different phases of schooling, but also prospective mathematics teachers have difficulties understanding the hierarchical classification of quadrilaterals ('House of Quadrilaterals'). The important role of the 'House of Quadrilaterals' in geometry curricula worldwide is not surprising, as "[t]he hierarchical classification of quadrilaterals might be regarded as an area of study which would help to promote the development of geometrical thinking" (Fujita, 2008, p. 31).

The active process of concept formation, required to build appropriate mental models of different quadrilaterals, and establish hierarchical relationships between them, is not trivial for learners due to cognitive complexities involved in its process (de Villiers, 1994; Monaghan, 2000). In order for mathematics teachers to foster adequate concept formation of the aforementioned topic in school geometry, specific professional knowledge is required. One the one hand, they need to possess content knowledge of the various quadrilateral concepts, and different hierarchical classifications of quadrilaterals. On the other hand, they also need to possess pedagogical-content knowledge on developing the 'House of Quadrilaterals' in the classroom, typical students' misconceptions, problematic circumstances regarding the lifelong learning of quadrilateral concepts and how these can be prevented, and models of learning geometric concepts. Given that the professionalization of prospective teachers is an essential task of teacher education programs, university courses for prospective teachers need to support prospective mathematics teachers in acquiring profound mathematical knowledge.

The present study aims to contribute to research endeavors dealing with the improvement of teaching and the professionalization of prospective primary school mathematics teachers within the university setting. Concretely, the objective of the study is to examine what professional knowledge on concept formation, using the example of the 'House of Quadrilaterals', prospective primary school mathematics teachers exhibited after attending a two-semester geometry course. Conceptualizing their understanding of the aforementioned topic across different professional knowledge domains is based on analysing their written answers from one knowledge test task using qualitative content analysis (Kuckartz, 2019; Mayring, 2014).

2 Theoretical background

The following section examines teacher professional knowledge, Weigand's model related to learning geometric concepts as well as research findings on the subject. The section ends with the three research questions that guided the study.

2.1 Professionalization of prospective mathematics teachers

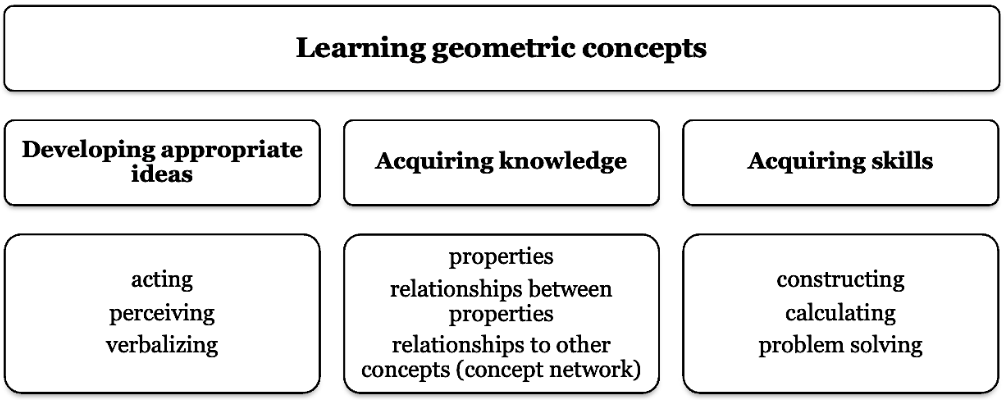
As one aspect of teacher competence, professional knowledge encompasses distinct areas of knowledge, such as content knowledge (CK) and pedagogical-content knowledge (PCK) Shulman's (1986). The former constitutes a form of knowledge that is grounded in the academic reference discipline, yet it represents a discrete area of knowledge. It is defined by curricula and exhibits an appropriate degree of subject matter understanding (Baumert & Kunter, 2011). Thus, mathematical CK refers to mathematics both as a discipline and a subject matter to be taught, with a focus on teachers' understanding of mathematical concepts, structures, and practices. According to Ball et al.'s (2008) model, CK comprises *common content knowledge* (CCK), which is relevant beyond teaching, and *specialized content knowledge* (SCK), which is limited to teaching. PCK concerns the teaching and learning (of mathematics) (i.e., explaining subject content in different ways, students' difficulties, providing individual support when learners encounter comprehension difficulties) (Baumert & Kunter, 2011). According to Ball et al.'s (2008) model, PCK comprises *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT).

Recent research (Dreher et al., 2018; Heinze et al., 2016) emphasizes *school-related content knowledge* (SRCK) as a theoretically justifiable and school-relevant area of subject knowledge that can be empirically distinguished from CK and PCK. SRCK "describes profession-specific conceptual subject knowledge about the connections between school and academic mathematics" (Heinze et al., 2016, p. 332). Thus, referring to reflection on the relevance of specific subject matter content and its relations, as well as the rationale for curricular decisions (Dreher et al., 2018).

2.2 Learning geometric concepts

An example of PCK, specifically related to knowledge about tasks and their curricular arrangement, is knowledge about the processes of and background to learning mathematical concepts. In order to understand a (geometrical) concept, it must be learnt, which is why this topic is relevant to all mathematics teachers and learners. According to Weigand's (2014) *model of learning geometric concepts*, the learning process consists of *developing appropriate ideas* about the concept, *acquiring knowledge* of its properties and relationships, and *acquiring skills* related to the concept. In teaching practices, these subprocesses are in constant interaction with one another, each consisting of individual activities (see Figure 1). According to Weigand (2014), appropriate geometric ideas — i.e., mental images of objects, their properties, and their relationships — "develop through actions on concrete objects, perceptions of objects and images, and descriptions or verbalizations of geometric objects" (p. 103). Properties, relationships between properties, and relationships to other concepts (concept network) represents necessary knowledge that should be acquired about any concept. A concept network can demonstrate the relationships between various forms of a single concept, as is the case with the 'House of Quadrilaterals', or illustrate the connections between different concepts (Weigand, 2015). Understanding the relationships between different quadrilateral concepts is important for developing an adequate mental model of the key concept 'quadrilateral' and its sub-concepts (e.g., square, rectangle), representing an important step in the concept learning process (Fujita, 2008). Finally, in order to understand a concept, it is important to acquire skills in operating with the concept, which includes constructing, calculating and problem solving.

Figure 1. Model of learning geometric concepts (Weigand, 2014)



2.3 Empirical findings on learners' understanding of the 'House of Quadrilaterals'

A number of studies reported that learners, both at school and university level, have difficulty understanding the 'House of Quadrilaterals' (de Villiers, 1994; Fujita, 2008; Fujita & Jones, 2007; Heinze, 2002; Kawasaki, 1992; Pickreign, 2007), namely with a hierarchical classification of quadrilaterals and, consequently, defining different quadrilaterals (Fujita & Jones, 2007). For example, the study results with 106 Grade 8 students showed that many students exhibited deficits in their concept understanding scheme of quadrilaterals (Heinze, 2002). Especially in cases in which students were required to apply different quadrilateral concepts to solve specific mathematical problems, the majority of students relied solely on their limited concept image of quadrilaterals, prohibiting solving problems successfully (Heinze, 2002). These results also supported de Villiers's (1994) findings, reporting that many students preferred a partitioned classification of quadrilaterals. Consequently, (prospective primary school) mathematics teachers must possess a profound content and pedagogical content knowledge of this area of mathematics to foster adequate students' understanding aligned with the mathematical perspective.

However, several studies (Fujita & Jones, 2007; Kawasaki, 1992; Pickreign, 2007) reported similar difficulties being experienced by prospective elementary school mathematics teachers. Kawasaki (1992) in his study with 52 Japanese prospective elementary teacher trainees reported that only a small sample (5%) could write a formal definition of a rectangle, and that many other participants used their own inappropriate concept image of a rectangle to define it. Similar results were reported by Pickreign (2007) in a study with 40 teacher trainees in the USA regarding their understanding of the properties of and relationships among parallelograms. In addition to previous findings, Fujita and Jones (2007) in their study with an opportunistic sample of trainee elementary school teachers (N = 263) enrolled in a four-year teacher training course in Scotland, reported on difficulties learners have with coming to an understanding of the hierarchical relationship between quadrilaterals. In conclusion, the aforementioned studies reported only implicitly on prospective teachers' professional knowledge, mainly addressing their content knowledge. Studies focusing on conceptualizing prospective mathematics teachers' understanding of the aforementioned topic across different professional knowledge domains have thus far not been conducted. Such research is of great importance in order to counteract the above-mentioned issues by targeting the development of prospective teachers' adequate professional knowledge of the topic.

2.4 Research questions

The following overarching research question guided the study: What professional knowledge did the prospective primary school mathematics teachers exhibit on concept formation in the 'Geometry and Teaching Geometry' course, using the example of the 'House of Quadrilaterals'? Concretely, the paper explores the following three research questions:

1. What concrete student (mis-)conceptions regarding the concepts of a 'square' and 'general quadrilateral' do the prospective primary school mathematics teachers adopt and identify?
2. What possible causes of such misconceptions do the prospective primary school mathematics teachers identify?
3. What scenarios for preventing the identified misconceptions, based on a model of learning geometric concepts, do the prospective primary school mathematics teachers develop?

3 Method

3.1 Research design and subjects

For this study, a mixed-methods research design was chosen in order to obtain findings that are as comprehensive and in-depth as possible. To record performance data, quantitative methods were employed. Additionally, qualitative content analysis (Kuckartz, 2019; Mayring, 2014) was used to allow interpretative evaluation of content-related, non-numerical data (Döring, 2023).

The study participants were prospective primary school mathematics teachers (Grades 1 to 6) who were in their second year of a teacher education program at a large German university. This group of prospective teachers was optimal for the purposes of the study, and was chosen because they take two geometry courses in the first semester ('Geometry and Teaching Geometry in Early Childhood Education' and 'Geometry and Teaching Geometry 1'), and a further geometry course in the second semester ('Geometry and Teaching Geometry 2') of their second year. Of the 128 prospective primary school mathematics teachers who took the course, 95 were included in the study.

3.2 Context of the study

This study was situated in one university initial teacher education department for prospective primary mathematics teachers in Germany. Two-semester 'Geometry and Teaching Geometry' course consisted of a 90' lecture and a 90' exercise on a weekly basis. In the course, subject-specific CK and PCK on school geometry was almost equally conveyed, and taught in close relation to each other. The exercises deepened the subject-specific CK and PCK conveyed in the lectures, in particular by using various analogue and digital learning manipulatives, providing teacher trainees with opportunities to experience different geometry school topics from the learner's perspective.

The topic of 'Concept formation' was covered in the first part of the course (winter semester), lasting four weeks. The content relevant to this study, namely the 'House of Quadrilaterals' and Weigand's (2014) model of learning geometric concepts, was discussed over the course of two weeks. The following was addressed in both lectures: definition of a concept, polygons and special quadrilaterals, the Van Hiele model of geometric thinking, typical problematic circumstances regarding the lifelong learning of quadrilateral concepts (i.e., prototypical representations, emphasis on differences between different quadrilaterals) and counteracting these, and Weigand's model of learning geometric concepts (see Sect. 2.2 & Figure 1). In the exercises, the prospective primary school mathematics teachers first discovered properties of different quadrilaterals from a perspective of a primary school student before developing one possible visualization of a hierarchical classification of quadrilaterals, and engaged in various learning stations addressing different aspects of the Weigand's model of learning geometric concepts.

The specific professional knowledge related to the 'House of Quadrilaterals' sought by the course can be summarized as follows: Prospective teachers should

- have an adequate understanding of the various quadrilateral concepts (CK: CCK, concept facet of SRCK);

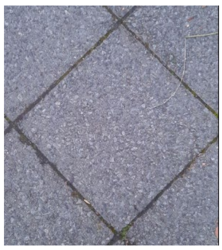
- be able to arrange different types of quadrilaterals in a local, hierarchical order (CK: SCK, concept facet of SRCK);
- be familiar with ways of developing the 'House of Quadrilaterals' in the classroom (PCK: KCT);
- be aware of problematic circumstances regarding the lifelong learning of quadrilateral concepts, and know how these can be avoided (PCK: KCS); and
- be familiar with Weigand's (2014) model of learning geometric concepts, and be able to design tasks pertaining to its different aspects (PCK: KCT).

3.3 Data collection instrument and data procedures

In order to obtain comprehensive and in-depth findings on the prospective primary school mathematics teachers' professional knowledge, a knowledge test was developed. The knowledge test covered a variety of primary school geometry topics (e.g., spatial visualization, symmetry, concept formation, tessellation, problem solving). One knowledge test task on the topic of 'concept formation', namely "Concept formation using the example of shapes" (see Figure 2), is of interest here. This task consisted of three sub-tasks, each addressing different facet(s) of professional knowledge (SRCK, PCK: KCS, KCT) needed for teaching the topic of concept formation in school geometry. In the following, a detailed description of each sub-task regarding the required professional knowledge facets is provided.

Figure 2. Knowledge test task on the topic of 'Concept formation'

Concept formation using the example of shapes (7 points)



Consider the following dialogue.
Merle: *"That's a square with the corner at the bottom."* (Points to one of the square paving stones on the pavement.)
Teacher: *"Isn't that also a quadrilateral?"*
Merle: *"No, a quadrilateral is straight at the bottom."*

a. Describe concisely and specifically what conception of a square and (general) quadrilaterals you recognize in Merle's statement. Explain concisely and specifically how Merle's ideas relate to the typical misconceptions you are familiar with. (2 points)
b. How could this misconception have arisen? Describe two possible causes concisely and specifically. (2 points)
c. How would you prevent this misconception? Describe two possible scenarios concisely and specifically, taking into account Weigand's model of learning geometric concepts (2014). (3 points)

Sub-task a. first required prospective teachers to adopt Merle's perspective based on her statements regarding a square and general quadrilateral. In order to successfully solve this sub-task, it is important to explicitly link subject-specific didactic content to common student errors and misconceptions, as well as to link the concept of a quadrilateral in a vertical manner within the spiral curriculum. In terms of professional knowledge, subject-specific PCK from the learner's perspective is required (PCK: KCS), as well as a general overview of the structure of quadrilateral-related subject CK in the spiral curriculum (SRCK: concept facet). Merle's statements underline her concept image of a (general) quadrilateral; quadrilaterals are 'straight at the bottom', i.e. they lie on their side and cannot have 'a corner at the top'. However, her concept image of a square is quite different. She recognizes the illustrated shape, and identifies it as a square, despite the form having 'a corner at the top'. These ideas should be related to known misconceptions in the second part of the sub-task a. From a pedagogical point of view, Merle has a rigid idea of the quadrilateral concept, as she only recognizes it in a prototypical representation ('lying on its side'). Furthermore, she does not yet seem to understand the relationship between the two concepts, so she does not yet grasp that 'quadrilateral' is a generic term for a 'square'.

Sub-task b. required prospective teachers to identify two possible causes for Merle's misconceptions. In order to successfully solve this sub-task, it is important to identify didactic difficulties along the path to a mathematically correct understanding of both concepts, and to be aware of typical misconceptions and their causes. In terms of professional knowledge, subject-specific PCK from the learner's perspective (PCK: KCS) is required. From a pedagogical point of view, several causes of Merle's underlying misconceptions are probable. Merle's misconceptions may be due to her lack of an adequate understanding of a general quadrilateral. It is also possible that she has mainly perceived, verbalized and dealt with prototypical representations of it, as these are usually illustrated 'side-on' and rarely take the shape of specific quadrilaterals in many school textbooks. Conversely, it is also plausible that Merle has not yet acquired sufficient knowledge of quadrilaterals, meaning the concepts of 'quadrilateral' and 'square' are not hierarchically connected in her concept image. Consequently, she does not recognize the square as a special case of a quadrilateral. Last but not least, it is possible that the characteristic properties of different types of quadrilaterals have been discussed, but no distinction has been made between superordinate and subordinate concepts.

Sub-task c. required prospective teachers to develop two teaching scenarios for preventing the identified misconceptions, referring to Weigand's (2014) model of learning geometric concepts. In order to successfully solve this sub-task, it is important to make an explicit reference to a subject-specific didactic model, as well as to discuss the use of analogue and digital materials needed to support the explanation. In terms of professional knowledge, subject-specific PCK of teaching through concrete models, illustrations, and the discussion of conceptual shifts (PCK: KCT) is required. From a pedagogical point of view, several teaching scenarios preventing the identified misconceptions are possible. For example, it is important that Merle first develops an appropriate understanding of both concepts; various less prototypical and dynamic representations of different quadrilaterals (including special ones) should be provided. Specifically, she should understand that quadrilaterals with four equal sides and four right angles are always squares, regardless of their position in space. It is also important that Merle learns to organize different quadrilateral concepts in a concept network, for example by using the 'House of Quadrilaterals', as well as to recognize that quadrilaterals with special properties are still quadrilaterals. In both learning scenarios, different analogous (geoboards, parallel strips), and dynamic tools can be used, to enable discovery learning and independent exploration of the relationships between the different quadrilateral concepts.

The aforementioned task was part of the final exam in the 'Geometry and Teaching Geometry' course. The prospective primary school mathematics teachers had three hours to complete the entire paper-based exam. Of the 103 prospective teachers registered for the final exam, 98 took part. Of these, 96 of them submitted at least one sheet for task 6, as required; however, one of these sheets was blank except for the prospective teacher's personal data. Therefore, a total of 95 submissions were examined for the purposes of the study.

3.4 Data analysis

After all the data were collected, the written documents from the knowledge test task were analyzed using descriptive statistics. Here, prospective teachers' answers to each sub-task were scored based on the developed scoring sheet. In the following step, the written documents were examined cross-sectionally using qualitative content analysis, driven by a seven-phase process (Kuckartz, 2019; Mayring, 2014). The initial screening of the data and the definition of main categories were guided by teacher professional knowledge facets (see Sect. 2.1). Specifically, the 95 written answers from the knowledge test were first digitized and anonymized to prevent conclusions being drawn about individuals. Following initial text work, and the writing of memos and case summaries needed to gain an overview, an initial deductive system of main categories was created based on the task structure (see Table 1). A separate code was assigned to each aspect of sub-task 6a. ('Merle's conception of a square', 'Merle's conception of a (general) quadrilateral', 'Merle's misconceptions'), which were then used to detect text passages relevant to RQ1. Similarly, with regard to RQ2, sub-task 6b. was assigned a code 'possible causes of misconceptions', as were both aspects of sub-task 6c. with regard to RQ 3 ('scenarios for preventing identified misconceptions' and 'reference to Weigand's model'). The latter procedure allowed recording the development of scenarios for preventing identified misconceptions and the reference to Weigand's (2014) model separately.

During the second coding process, subthemes were formed inductively from the existing data to structure the content of the statements in accordance with the research questions. This resulted in a hierarchically organized category system which is exemplary explained with regard to RQ2 by examining one prospective teacher's written statement *"Merle thinks that a square has four sides of equal length and four right angles"*. In the first coding stage, it was categorized under the main category '6a.1 Merle's conceptions of a square', and subsequently developing the subtheme 'Squares have four sides of equal length and four right angles'. During further coding, the statement *"Squares, on the other hand, seem to be a shape with specific properties that the square paving stone fulfils despite rotation"* was used to create another subtheme, namely 'Square as a shape with specific properties', containing more general statements about Merle's conceptions of a square. This resulted in the following hierarchy: '6a.1 Merle's conceptions of a square' > 'Squares have four sides of equal length and four right angles' > 'Square as a shape with certain properties'.

Table 1. Main qualitative content analysis' categories of the knowledge test task in relation to the research questions

Main categories		**Research questions**
6a.1	Merle's conception of a square	RQ1: What concrete student (mis-)conceptions regarding the concepts of a 'square' and 'general quadrilateral' do the prospective primary school mathematics teachers adopt and identify?
6a.2	Merle's conception of a (general) quadrilateral	
6a.3	Merle's misconceptions	
6b.	possible causes of misconceptions	RQ2: What possible causes of such misconceptions do the prospective primary school mathematics teachers identify?
6c.1	scenarios for preventing identified misconceptions	RQ3: What scenarios for preventing the identified misconceptions, based on a model of learning geometric concepts, do the prospective primary school mathematics teachers develop?
6c.2	reference to Weigand's model	

Utilizing both deductive and inductive coding processes resulted in a very comprehensive and consistent category system which contained detailed descriptions of main themes and subthemes, accompanied with illustrative quotations in the form of anchor examples. This allowed the coding to be traced, so that auditability can be considered a given (Döring, 2023). Furthermore, utilizing inductive coding processes allowed flexibility and openness sought in qualitative research. By embedding the knowledge test task in the final exam in the 'Geometry and Teaching Geometry' course, the study participants were not aware while completing the exam that their answers served a purpose beyond performance assessment for the module grade. Therefore, potential interactions between the researcher and the study participants could not influence their experience and behavior, and non-reactivity can be assumed (Döring, 2023).

4 Results

In this section, results pertaining to the three research questions, namely adopting Merle's conception of a square and general quadrilateral and identifying underlying misconceptions (RQ1), identifying possible causes of misconceptions (RQ2), and developing scenarios for preventing the identified misconceptions in connection to Weigand's model of learning geometric concepts (RQ3), are presented.

Before focusing on each research question, a brief descriptive statistic on the knowledge test task on concept formation is provided. Of the 95 prospective teachers, two did not complete sub-task b. and sub-task c. Figure 3 illustrates the distribution of the points achieved on the knowledge test task. Two prospective teachers (2.1%) achieved less than one point, four (4.2%) achieved the full seven points, and the majority of them (27.4%) achieved at least five but less than six points. A more detailed evaluation of the distribution of the points on the knowledge test task is provided in Table 2.

Figure 3. Grouped frequency distribution of points achieved on the knowledge test task on concept formation

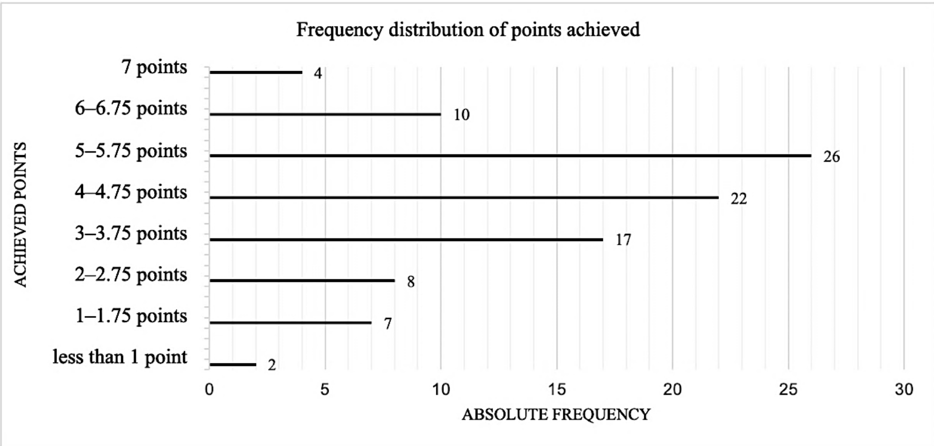


Table 2. Distribution of the points achieved on the knowledge test task

Task	n	Min	Max	M	Scoring
6a	95	0	2	1.19	2
6b	93	0	2	1.22	2
6c	93	0.5	3	1.99	3
Total	95	0	7	4.29	7

4.1 Adopting Merle's conception of a square and general quadrilateral and identifying underlying misconceptions (RQ1)

4.1.1 Adopting Merle's conception of a square and general quadrilateral

The dialogue between Merle and the teacher in the sub-task a. (see Figure 2) indicates that Merle recognizes one of the square paving stones on the pavement, and identifies it as a square, despite its position (i.e., 'corner at the bottom'). Merle's conception of a square adopted by the prospective teachers is illustrated in Table 3 in which data-driven themes and subthemes with anchor examples are provided. The majority of prospective teachers could not adopt Merle's perspective with regard to her concept image of a 'square' as their answers were vague (Theme 4; n = 57). This also applies to prospective mathematics teachers who described Merle's concept of a square as dependent of its position in space/non-rotatable (Theme 1; n = 18). Here, the dialogue was misinterpreted as such conception of a square could not be logically derived from the task. Furthermore, there were a few instances of overinterpretation, some of which revealed prospective teachers' misconceptions, such as "*A square is also a rhombus and can therefore also have 'a corner at the bottom'*". The second, problematic part of this sentence reflects a prototypical concept image of both rhombus and square, and thus inadequate concept images of both shapes.

Table 3. Themes related to adopting Merle's conception of a square

Theme 1: Dependent on the position in space/non-rotatable (n = 18) (contains descriptions of Merle's concept of a square that is dependent of its position in space/non-rotatable, as well as more specific subthemes)	
stands on one corner	"Merle assumes that … its sides run diagonally."
lies on one side	"She thinks that a square always lies on one side."
Theme 2: Independent of the position in space/rotatable (n = 7) (contains descriptions of Merle's concept of a square that is independent of its position in space/rotatable, as well as more specific subthemes)	
does not have to lie on one side	"Merle assumes that in a quadrilateral, one of the sides must always be at the bottom. In her view, this doesn't apply to a square."
flexible use of a concrete image	"This dialogue shows that Merle has a clear idea of what a square should look like and can apply this idea flexibly, as in the example by rotating it."
both a side and a corner can be at the bottom	"The square is probably known in two versions, in the form shown and rotated by 45°."
non-prototypical image	"Interestingly, she doesn't have this [an 'ideal prototypical idea'] of a square, even though the square depicted is not shown in its prototypical arrangement."
Theme 3: Shape with specific properties (n = 13) (contains descriptions of Merle's conception of a square as a shape with certain properties, as well as more specific subthemes)	
four right angles	"Because here, too, the four 90° angles would be present."
corners	"Merle's statement shows that she knows that a square has corners."
four sides of equal length	"So, she already recognizes the property of four sides being of equal length."
four sides of equal length and four right angles	"Merle imagines that a square has four sides of equal length and four right angles."
Theme 4: Lack of a concrete description (n = 57) (applies when Merle's conception of a square is not described in concrete terms)	
	"Merle already recognizes different shapes and can name them. However, she only recognizes them when they are presented in the position she has learned."

Few prospective teachers (n = 7) adopted correctly Merle's concept of a square as independent of its position in space/rotatable (Theme 2) by referring, for instance, to her non-prototypical image of a square or flexible use of a concrete image. Some prospective teachers' descriptions (n = 13) adopted Merle's conception of a square as a shape with certain properties (Theme 3), such as *"Squares, on the other hand, appear to be only shapes with certain properties that the stone fulfills despite rotation"*. Thus, these prospective teachers adopted another perspective, namely Merle's holistic understanding of a square based on the shape's properties, enabling her to identify one of the square paving stones on the pavement as a square.

The dialogue between Merle and the teacher (see Figure 2) indicates that Merle possesses a rigid idea of the quadrilateral concept, as she only recognizes it in a prototypical representation ('lying on its side'). Merle's conception of a general quadrilateral adopted by the prospective teachers is illustrated in Table 4 in which data-driven themes and subthemes with anchor examples are provided. The majority of prospective teachers (n = 52) adopted correctly Merle's concept of a general quadrilateral as dependent of its position in space/non-rotatable (Theme 1) by referring to her image of a general quadrilateral in which one side is always at the bottom. However, it has also been described that Merle sees quadrilaterals as a more specific quadrilateral (i.e., kite/rhombus, square/rectangle, rectangle) which could not be derived from the task in a logically consistent manner. Almost half of the prospective teachers (n = 38) could not adopt Merle's perspective with regard to her concept image of a general quadrilateral as their answers were vague (Theme 3).

Table 4. Themes related to adopting Merle's conception of a general quadrilateral

Theme 1: Dependent on the position in space/non-rotatable (n = 52) (contains descriptions of Merle's concept of a general quadrilateral that is dependent of its position in space/non-rotatable, as well as one specific subtheme)	
lies on one side	"Merle assumes that in a quadrilateral one side is always at the bottom."
Theme 2: Another quadrilateral (n = 5) (contains descriptions of Merle's conception of a general quadrilateral as a more specific quadrilateral, as well as more specific subthemes)	
kite/rhombus	"Merle's image of a quadrilateral is of a kite or rhombus."
square/rectangle	"When Merle hears the term 'quadrilateral,' she imagines a classic square or rectangle."
rectangle	"And by quadrilaterals, she only means rectangles, since she doesn't answer 'yes' to the question."
Theme 3: Lack of a concrete description (n = 38) (applies when Merle's conception of a general quadrilateral is not described in concrete terms)	
	"Merle doesn't have a clear understanding of the quadrilateral content."

4.1.2 Identifying underlying misconceptions

In the second part of the sub-task a., the identified Merle's conceptions of a square and general quadrilateral needed to be related to known misconceptions which include both Merle's prototypical, rigid image of the quadrilateral concept (i.e., as long as the square stands on its corner, it cannot be a quadrilateral, prototype of the quadrilateral prevails), and the lack of understanding of the relationship between the two concepts (i.e., 'quadrilateral' is a superordinate term of 'square'). Both of these misconceptions were identified in the written answers as illustrated in Table 5, which provides data-driven themes and subthemes with anchor examples. Especially, the latter was recognized by the prospective teachers as all written answers reflected this notion (Theme 5). Here, almost all prospective teachers' answers (n = 92) were concrete by describing a missing understanding of superordinate and subordinate concepts in the context of quadrilaterals or relationship between quadrilaterals and squares. The second possible underlying misconception, namely inadequate understanding of the shape's appearance and position in space as a property of a shape (Theme 4) was also described (n = 35) by referring, for instance, to Merle's prototypical images of a square and/or quadrilaterals and/or shapes in general. Here, drawing on her knowledge of a particular representative based on such illustrations in the school textbooks, was particularly emphasized. In case of Themes 1 to 3, possible causes of misconceptions were described (n = 14) rather than Merle's misconceptions.

Table 5. Themes related to identified underlying misconceptions

Theme 1: Incorrect properties noted (n = 1) (contains descriptions of Merle's misconception insofar she has incorrectly memorized the properties)	
	"She has memorized and internalized false properties."
Theme 2: Basic conceptual understanding not sufficient (n = 10) (contains descriptions of Merle's misconception insofar her understanding of the concept is insufficient or inadequate, as well as more specific subthemes)	
non-flexible knowledge	"So, she isn't yet linking her knowledge. She cannot apply her knowledge flexibly."
unknown properties of the figures	"She doesn't yet know the properties of quadrilaterals."
Theme 3: Non-focused properties (n = 3) (contains descriptions of Merle's misconception insofar she does not focus on the properties)	
	"In defining the term, she didn't take into account the properties of the figure."
Theme 4: Unclear idea of appearance and location in space (n = 35) (contains descriptions of Merle's misconception insofar her understanding of a shape's appearance and position in space is not adequate, as well as more specific subthemes)	
dependent on the position in space	"Merle has the misconception that spatial orientation is part of it and doesn't know that shapes can be rotated."
rigid/static/unflexible image (of quadrilaterals, of squares)	"There is an unflexible image about the square; for Merle, the square has a certain appearance and cannot be changed."

prototypical image (implicitly, explicitly)	"For her, shapes only have a certain appearance. So, she has a misconception because she draws on her knowledge of a particular representative."
Theme 5: Relationships between quadrilaterals unclear/unknown (n = 95) (contains descriptions of Merle's misconception insofar relationships between different quadrilaterals are still unclear or unknown, as well as more specific subthemes)	
superordinate and subordinate concepts not understood/known	"Merle's images are very similar to the misconceptions I am familiar with, as she isn't yet able to properly distinguish between terms and generic terms and lacks the correct specific properties for the correct definition."
relationship between square and quadrilateral not known/recognized	"Merle has the misconception that squares and quadrilaterals are not related, because if she knew that a square is a quadrilateral, she would have answered the teacher's question with 'yes'."
other types of quadrilaterals (off topic)	"The misconception here is that she doesn't yet understand the relationship between squares and rectangles."

Based on both Table 3 and Table 4, it can be summarized that at the time of the knowledge test, the prospective teachers could only partially adopt Merle's perspective, representing a specific student, with regard to her concept images of a 'square' and 'general quadrilateral'. Though all prospective teachers deduced one existing misconception, the majority was not able to fully deduce other possible misconceptions. Thus, it can be concluded that the prospective primary school mathematics teachers exhibited deficits in their professional knowledge (PCK: KCS and SRCK: concept facet) in the context of the sub-task a., though this image was not unanimous.

4.2 Identifying possible causes of misconceptions (RQ2)

The sub-task 6b. focused on identifying two possible causes of the misconceptions identified in the sub-task 6a. Firstly, Merle's misconception could be due to the fact that she has mainly been presented with prototypical representations of a (general) quadrilateral. Secondly, it could be that the concepts of a general quadrilateral and square were not linked, so that the square was not recognized as a special case of a quadrilateral. Accordingly, characterizing properties were probably emphasized, but either no distinction was made between superordinate and subordinate concepts or similarities between different concepts may not have been discussed. Possible causes of Merle's misconceptions identified by the prospective teachers are illustrated in Table 6 in which data-driven themes and subthemes with anchor examples are provided. The aforementioned causes as well as other possible causes of misconceptions were organized around three themes: possible causes that are 'not Merle's responsibility' (n = 162), possible causes that are 'Merle's responsibility' (n = 28), and '(re)described misconceptions' (n = 4) without mentioning a potential cause. In the following, the first two themes are presented in more detail.

Looking across prospective teachers' written answers pertaining to Theme 1 (n = 162), different aspects of inadequate concept formation were expressed. In addition to aforementioned lack of different representatives, and unknown relationship between the shapes, introducing the shapes by solely focusing on explicitly definitional concept formation were classified as potentially causative. Here, however, also inadequate and non-potential causes were identified, such as ‘too little problem solving’ and ‘properties not sufficiently known’. In one case, a prospective teacher wrote that Merle’s misconception could be due to confusion caused by the introduction of the square. This reasoning is potentially causative: different quadrilaterals (general quadrilateral, square, rectangle) are generally introduced at an early age by the term quadrilateral before differentiating between them. Other causes (i.e., synonymous use of terminology, little/no connection to everyday life, level of treatment of quadrilateral types) were classified as not potentially causative for Merle’s presumed misconceptions. Possible causes were also mentioned explicitly referring to Merle (Theme 2; n = 28), reflecting, for instance her inability of mental rotation, lack of mental imagery. All of the aspects for which Merle would be responsible were classified as not logical potentially causal, as the sub-task 6b. focused on teaching geometry. For that reason, just one specific subtheme is listed.

Table 6. Themes related to possible causes of misconceptions

Theme 1: not Merle’s responsibility (*n* = 162) (contains subthemes of the possible causes mentioned for Merle’s misconceptions, which were not attributed to Merle’s responsibility)	
inadequate concept formation (explicitly definitional concept formation, relationship between shapes unknown, few different representatives covered, properties not sufficiently known, not experienced enactively/actively, too little problem solving)	“This happens, for example, when concepts are introduced solely on the basis of their definition: X is such-and-such, and Y is such-and-such. This corresponds to the explicit-definitional type of concept formation.”
confusion caused by the introduction of the square	“Merle may also have been confused by the terminology square, since the shape is also a quadrilateral and, before the term square was introduced, it had always been a quadrilateral to her. She cannot cope with two terms for which she sees as two different properties.”
synonymous use of square and quadrilateral terminology	“In everyday language, the terms square and quadrilateral are often used synonymously.”
little/no connection to everyday life/the environment	“... lack of connection to the environment.”
different levels of treatment of quadrilateral types	“Alternatively, it is also possible that she gained additional experience with squares, while quadrilaterals were only treated marginally.”
Theme 2: Merle’s responsibility (*n* = 28) (contains subthemes of the possible causes mentioned for Merle’s misconceptions, which were attributed to Merle’s responsibility) *here one subtheme only is presented	
mental rotation not used	“On the other hand, she may not be aware that a quadrilateral can also be rotated so that the corners can always be at different viewing angles.”
Theme 3: Misconception (re)described without naming the cause (*n* = 4) (contains descriptions of possible causes for Merle’s misconceptions, which are not actually causes at all, but (re-described) misconceptions)	
	“There are general misconceptions about relationships between shapes, and shapes and their properties.”

In summary, for various misconceptions identified in the sub-task a., their different potential causes were either classified as potentially causative for Merle's presumed misconceptions in accordance with the pedagogical perspective (n = 98 concrete, n = 52 non-concrete) or not (n = 40). Thus, it can be concluded that the prospective primary school mathematics teachers exhibited deficits in their professional knowledge (PCK: KCS) in the context of the sub-task b., though this image was not unanimous.

4.3 Theory-driven development of scenarios for preventing the identified misconceptions (RQ3)

4.3.1 Scenarios for preventing the identified misconceptions

The sub-task 6c. focused on developing two scenarios for preventing the identified misconceptions in the sub-task 6a. As previously noted, to prevent Merle's misconceptions, it is of importance to establish an adequate concept image of the quadrilateral concept. For instance, various, less prototypical representatives of different quadrilaterals, especially varying in their position in space, could be provided which can then be perceived, verbalized, or dealt with actively by using appropriate materials (e.g., geoboards, DGS). Additionally, Merle's misconceptions could be further prevented by scenarios that would allow developing a relationship between a square and quadrilateral. Scenarios for preventing the identified misconceptions developed by the prospective teachers are illustrated in Table 7 in which data-driven themes and subthemes with anchor examples are provided. The aforementioned scenarios as well as other scenarios, appropriate and goal-oriented, not necessarily goal-oriented, vague and inappropriate ones, were organized around 13 themes which are presented in the following.

Scenarios in which diverse representatives (Theme 1) are used, were suggested by many prospective teachers (n = 77), though the quality of the written answers varied greatly. Besides appropriate and goal-oriented suggestions, such as illustrating or bringing many different quadrilaterals, some were very vague without specifying the shapes or inappropriate as the shapes mentioned, were not part of the task (i.e., rectangle) or were not connected to Merle's misconceptions (i.e., square and rectangle). Avoiding typical prototypical representations (Theme 2) by developing own worksheets instead of working with textbooks that indulge prototypical representations was mentioned by one prospective teacher only. The majority of written answers contained descriptions of appropriate and goal-oriented scenarios involving the inclusion of different types of quadrilaterals at an early age (Theme 3; n = 91), specifically by examining the relationship between different quadrilaterals. Here, concrete references pertaining to examining the relationship between a square and quadrilateral or between different quadrilaterals by focusing on similarities between the shapes or class inclusion were made.

Alternatively, scenarios were suggested that involved working with different materials (Theme 4; n = 65), namely enactive, static and dynamic. Scenarios with static materials (e.g., laying with pattern blocks) are only partially effective as they retain a static character.

Yet, *"Through concrete material that can be turned and flipped"* students would be provided with visualizations of quadrilaterals in different positions in space, thus prohibiting the development of a prototypical image. Nevertheless, suggestions were made that were neither effective (i.e., cutting shapes out of paper) nor appropriate (i.e., drawing shapes). Besides working with static materials, scenarios containing the use of dynamic materials were described, such as using the drag mode in a digital geometry software (GeoGebra), spanning on the geoboard or working with parallel stripes. Such dynamic scenarios are appropriate and goal-oriented as they allow the square to be "discovered" as a special case of a general quadrilateral. However, some scenarios were inappropriate as they focused on the relationship between a square and rhombus, or were too vague to be assessed as no concrete shapes were mentioned.

Sensible and goal-oriented suggestions of scenarios focusing on environmental reference (Theme 5; n = 17), defining the concepts genetically (Theme 6; n = 1), and exemplary concept formation (Theme 7; n = 1) were also made. The suggestions regarding to former were mainly concrete as the example in Table 7 shows. However, one prospective teacher suggested students "*could collect four-angled packaging and bring it with*", which would be inappropriate as packaging is tree- dimensional. This example clearly reflects inadequate PCK which may be based on inadequate CK. The latter two themes were very vague in concretely explaining how these would prevent the identified misconceptions. Within Theme 6, three proposals of defining the concepts with real definitions were made. However, such scenario is considered to be inappropriate, given that such definitions can be understood by the students when the understanding of the relationships between figures is given. In rare cases, the scenarios included creating letter profiles for characteristics, use of various types of paper, creating a lapbook, encourage exchange between students, and teacher playing a central role (Themes 8-12), which were inappropriate and/or too vague. Finally, several answers (n = 12) did not contain any formulation of a specific scenario (Theme 13).

Table 7. Themes related to scenarios for preventing the underlying misconceptions

<table>
<tr><td colspan="2">Theme 1: Diverse representatives (n = 77)
(contains descriptions of scenarios for preventing Merle's misconceptions by including diverse representatives, as well as more specific subthemes)</td></tr>
<tr><td>of general quadrilaterals</td><td>"I would present or bring along many different representatives of quadrilaterals."</td></tr>
<tr><td>of quadrilaterals and squares</td><td>"... create different quadrilaterals or squares."</td></tr>
<tr><td>of squares</td><td>"... very different representatives of squares."</td></tr>
<tr><td>vague</td><td>"... various representatives are presented and compared."</td></tr>
<tr><td>of squares and rectangles</td><td>"To correct this misconception, I would show Merle examples of squares and rectangles from her surroundings."</td></tr>
<tr><td>of quadrilaterals and rectangles</td><td>"I would ask my students to fold, draw and bend various rectangles and quadrilaterals."</td></tr>
<tr><td colspan="2">Theme 2: Avoid prototypical representations (n = 1)</td></tr>
</table>

(contains descriptions of scenarios for preventing Merle's misconceptions by avoiding prototypical representations)	
	"Furthermore, I would avoid using textbooks with standard constructions as much as possible and also create my own worksheets with non-standardized quadrilaterals."
Theme 3: Inclusion of different types of quadrilaterals at an early stage (*n* = 91) (contains descriptions of scenarios for preventing Merle's misconceptions through the early inclusion of different types of quadrilaterals, as well as one specific subtheme)	
Relationships between different quadrilaterals (square-quadrilateral, comparing different quadrilaterals)	"They should then compare them [different shapes] and discuss their similarities and differences."
Theme 4: Working with materials (*n* = 65) (contains descriptions of scenarios for preventing Merle's misconceptions using unspecified material, as well as more specific subthemes)	
enactive	"I would primarily work enactively with shapes/figures."
static material (folding/cutting paper, laying with pattern blocks)	"The children receive rectangles, squares, etc. as tiles and can turn them, flip them, and look at them."
dynamic material (DGS, spanning on geoboard, parallel stripes)	"I would demonstrate on geoboard how by shearing the square so that the four sides remain equal in length, but the angles changes, a rhombus is formed."
Theme 5: Environmental reference (*n* = 17) (contains descriptions of scenarios for preventing Merle's misconceptions by referring to the environment)	
	"… perceiving squares and quadrilaterals in the environment, lots of different ones."
Theme 6: Explicit definition of concepts (*n* = 5) (contains descriptions of scenarios for preventing Merle's misconceptions by means of defining concepts, and more specific subthemes)	
real definitions	"The terms could be introduced with real definitions."
genetic definitions	"… genetically define …"
Theme 7: Exemplary concept formation (*n* = 1) (contains descriptions of scenarios for preventing Merle's misconceptions by means of exemplary concept formation)	
	"… exemplary concept formation."
Theme 8: Letter profile for characteristics (*n* = 5) (contains descriptions of scenarios for preventing Merle's misconceptions by means of letter profiles for characteristic)	
Theme 9: Various types of paper (*n* = 1) (contains descriptions of scenarios for preventing Merle's misconceptions by using various types of paper Anchor example:	
	"I would provide the learner with graph paper, dot paper, and isometric paper)."
Theme 10: Create a lapbook (*n* = 2) (contains descriptions of scenarios for preventing Merle's misconceptions by creating lapbooks)	
Theme 11: Exchange between students (*n* =2) (contains descriptions of scenarios for preventing Merle's misconceptions through exchange between students)	
	"In addition, students should exchange ideas with each other."
Theme 12: Central role of the teacher (*n* = 2)	

(contains descriptions of scenarios for preventing Merle's misconceptions in which the teacher plays a central role)	
	"... [me as a teacher] by drawing and verbalizing the house of quadrilaterals."
Theme 13: No specific scenario formulated (*n* = 12) (contains examples in which no specific scenario was formulated)	

In summary, the theory-based scenarios for preventing the misconception identified in sub-task c. could be classified as 'appropriate and goal-oriented' (n = 133), 'not necessarily goal-oriented' (n = 51), 'too vague to be assessed' (n = 73), 'inappropriate' (n = 12), according to the pedagogical perspective, or no scenario was formulated (n = 12). Though all prospective teachers described one possible scenario, the majority was not able to describe other appropriate and goal-oriented scenarios. Thus, at the time of the knowledge test the prospective primary school mathematics teachers could only partially formulate suitable, goal-oriented scenarios to prevent Merle's misconceptions.

4.3.2 Reference to Weigand's model of learning geometric concepts

As a part of the sub-task c., the developed scenarios needed to be connected to Weigand's model of learning geometric concepts. Given Merle's misconceptions, the developed scenarios should address the development of appropriate understandings of both concepts as well as knowledge acquisition, specifically relationships to other concepts (conceptual network). References to Weigand's model of learning geometric concepts made by the prospective teachers are illustrated in Table 8 in which data-driven themes and subthemes with anchor examples are provided. The aforementioned references as well as other ones were organized around 5 themes which are presented in the following.

The majority of the prospective teachers linked their scenarios to aforementioned aspects of Weigand's model, namely developing appropriate ideas (Theme 1; n = 47), and acquisition of knowledge (Theme 2; n = 43). Here, besides identifying the main aspects of the model, sub-aspects were also mentioned. Whereas perceiving and relationships to other concepts (concept network) are appropriate and goal-oriented, other sub-aspects, such as verbalizing, acting, and properties are not effective in isolation. References were also made to the 'acquisition of skills' (Theme 3; n = 20) which is not relevant for the context, and thus inappropriate. However, references to the model were made that were inappropriate from the pedagogical perspective. For example, one prospective teacher wrote: *"Like Weigand, I would introduce the concepts of squares and quadrilaterals by drawing and discussing the 'House of Quadrilaterals'."* This statement reflects this prospective mathematics teacher's inadequate understanding of Weigand's model of learning geometric concepts. To her or him it is not clear that mere drawing of a shape without discussing each shape's properties may foster students' prototypical images of a particular shape. Furthermore, by just saying "discussing the 'House of Quadrilaterals'" would be done, shows that this prospective teacher does not possess the knowledge of how to foster students' understanding of the complexity of the relationships between different shapes.

Finally, some prospective teachers made no reference or only a vague reference to Weigand's model (Theme 4; n = 23) whereas others only mentioned aspects of the model without linking them to a specific scenario (Theme 5; n = 15).

Table 8. Themes related to references to Weigand's model of learning geometric concepts

Theme 1: Developing appropriate ideas (n = 47) (contains scenario-linked mentions of the model aspect of developing appropriate ideas, as well as more specific subthemes)	
perceiving	"Practicing recognizing quadrilaterals and squares in the environment."
verbalizing	"Verbalizing."
acting	"Acting."
Theme 2: Acquisition of knowledge (n = 43) (contains scenario-linked mentions of the model aspect of knowledge acquisition, as well as more specific sub-themes)	
properties	„Properties."
relationship between properties	„Relationship between properties."
relationship to other concepts	"Relationships of the individual quadrilaterals."
Theme 3: Acquisition of skills (n = 20) (contains scenario-linked mentions of the model of skill acquisition, as well as more specific subthemes)	
constructing	"Constructing."
calculating	"Calculating.
problem solving	"Problem solving."
Theme 4: Vague/No reference to model (n = 23) (contains descriptions of scenarios that do not refer to any model or are vague)	
	"Explain that figures can be transformed and are free from orientation in space."
Theme 5: Reference to model without linking to scenario (n = 15) (contains references to one or more aspects of the model without the context of a specific scenario)	
	"These misconceptions can be prevented by developing appropriate understandings. This means verbalizing concepts, perceiving them, and acting on them accordingly."

In summary, at the time of the knowledge test the prospective teachers could only partially connect their developed scenarios to prevent the potential causes of Merle's misconceptions by referring to suitable aspects of the Weigand's model of learning geometric concepts. Thus, it can be concluded that the prospective primary school mathematics teachers exhibited profound deficits in their professional knowledge (PCK: KCT) in the context of the sub-task c.

5 Discussion

The final section first discusses prospective primary school mathematics teachers' professional knowledge of the 'House of Quadrilaterals'. This is then followed by considering study limitations, along with some possible future research directions and practical implications for mathematics teacher education.

5.1 Prospective primary school mathematics teachers' professional knowledge of the 'House of Quadrilaterals'

In light of the overarching research question "What professional knowledge did the prospective primary school mathematics teachers exhibit on concept formation in the 'Geometry and Teaching Geometry' course, using the example of the 'House of Quadrilaterals'?, the findings can be summarized as follows. The quantitative data revealed that only a portion of prospective primary school mathematics teachers exhibited the specific professional knowledge required for concept formation in the case of the 'House of Quadrilaterals'. Whereas the majority of the prospective primary school mathematics teachers achieved more than half of the points, many also received very few. The qualitative data revealed that many prospective primary school mathematics teachers had at least a basic understanding of the concepts of a square and general quadrilateral, and their relationship to each other (PCK: KCS). A similar picture arose regarding their subject-specific knowledge of the structure of quadrilateral-related content in the spiral curriculum (SRCK: concept facet) from a learner's perspective. However, some also faced considerable difficulties in the aforementioned areas of professional knowledge, conferring the previous findings on prospective teachers' misconceptions (Fujita & Jones, 2007; Kawasaki, 1992; Pickreign, 2007). The misconceptions connected to an image of a particular quadrilateral as well as relationships between different quadrilaterals, clearly imply the lack of basic CK every prospective mathematics teacher should possess. A similar picture arose with regard to knowledge of didactic difficulties on the way to a mathematically correct understanding of the aforementioned concepts, typical misconceptions and their causes (PCK: KCS), and subject-specific PCK from the learners' perspective in connection to Weigand's (2014) model of learning geometric terms (PCK: KCT). Especially, not-goal oriented written answers reflected prospective mathematics teachers' inadequate understanding of the Weigand's model of learning geometric concepts.

Though some prospective teachers' written answers were suitable, concrete and goal-oriented, many vague answers could be explained by lack of professional knowledge on content formation, time pressure, nervousness, test anxiety or difficulty understanding the task. The exhibited deficits in integrating the necessary professional knowledge across all knowledge facets on concept formation using the example of the 'House of Quadrilaterals' are alarming given that this topic is an integral part of geometry curricula, and an important area of study that helps foster geometrical thinking (Fujita, 2008).

5.2 Limitations of the study and future research directions

This study was an exploratory mix-methods study using purposive sampling, namely prospective primary school mathematics teachers who attended the two-semester 'Geometry and Teaching Geometry' course in one university initial mathematics teacher education department in Germany, and wrote the final exam at the end of the course. Thus, the findings cannot be transferred to all prospective primary school mathematics teachers, since the sample was not randomly chosen, but is rather illustrative of other similar samples. These limitations implicate a possible next step in research, namely to conduct a study with a larger data sample in a wider variety of settings (e.g., different mathematics teacher education programs), and use alternative sampling strategies (e.g., probability sampling). This would allow a more thorough understanding of prospective primary school mathematics teachers' professional knowledge which can then be generalizable to a population.

Given the study design, only prospective primary school mathematics teachers' professional knowledge on concept formation, using the example of the 'House of Quadrilaterals', was investigated. To achieve this goal, one data source only, namely written responses on one knowledge test task, was used. These limitations suggest a possible next step in research, namely to investigate the overarching research question in more detail, by broadening the content context (e.g., other geometry concepts), and by focusing on all aspects of teacher professional knowledge. Concretely, in an interview context, prospective teachers could be asked about their concept images of a particular geometric concept (CK: CCK), and then inquire more about the possible underlying conceptions, mental barriers and difficulties when discussing different interview stimuli.

In the study, the prospective teachers' professional knowledge was examined after completing the two-semester course, though the topic of concept formation was dealt with in the first part of the course. To examine the immediate effect of the course on the development of professional knowledge on concept formation using the example of 'House of Quadrilaterals', prospective mathematics teachers could be surveyed before and directly after addressing the topic. Such study design could be supplemented by a follow-up examination at the end of the semester or as part of the knowledge test, if necessary. In that manner a researcher could create a more thorough description of the development and acquisition of professional knowledge as well as to make implications regarding the sustainability of the course.

The setting of study context may have also influenced the findings, due to time pressure, nervousness, and exam anxiety. A similar task could alternatively be administered in the context of a weekly homework assignment. This would have several advantages; for one, prospective teachers would not be under time pressure, and exam anxiety would not be a disruptive factor. Furthermore, they would have access to all the necessary course material and could exchange ideas with other peers. Methodologically, data collection would be much timelier. However, data collected in such a setting would probably be more conducive to drawing conclusions about the prospective teachers' underlying professional knowledge.

5.3 Practical implications for mathematics teacher education

By relating the study results to teaching practice, some implications for mathematics teacher education can be drawn. Disregarding external factors (e.g., time pressure, exam anxiety), the difficulties experienced by some prospective teachers are most likely due to their own misconceptions, which is supported by earlier studies (Fujita & Jones, 2007). Given that underlying subject-specific CK (CCK and SCK) determines subject-specific PCK (KCS and KCT), as well as the SRCK (concept facet) (Dreher et al., 2018; Heinze et al., 2016), conception or re-design of education courses for prospective mathematics teachers should focus on strengthening the relevant mathematical foundation. Examination of the ‘Geometry and Teaching Geometry’ course content on the topic of concept formation revealed that the prospective teachers were offered different learning opportunities at both the content and process levels (see Sect. 3.2). This raises many questions, such as the question of their school geometry experiences with the different quadrilateral concepts, question of sufficient time being given in the course dealing with the topic (dependent on their school experiences), question of their ability to acquire both CK and PCK at the same time, especially when knowledge gaps from or developed in school geometry lessons exist.

The new developments in teacher education call for (re)designing (existing) courses that would attend to specific design principles in order to support the development of prospective teachers’ adequate professional knowledge (Reitz-Koncebovski et al., 2022), such as ‘pedagogical biplane’. The ‘pedagogical biplane’ is a teaching methodology (Wahl, 2013) that focuses on repeatedly engaging prospective teachers on a meta-level alongside the teaching of subject-specific or didactic content. Here questions such as “What are we doing here and why are we doing it?”, “What can you learn from this for your future career as a mathematics teacher?” have been proven beneficial for the effectiveness of the mathematics education courses (Reitz-Koncebovski et al., 2022). For instance, in the ‘Geometry and Teaching Geometry’ course it would be possible to present the structure of the ‘House of Quadrilaterals’ across the various grades in accordance with the spiral curriculum or to ask prospective teachers to develop possible representations of it and reflect on these. This would allow a stronger focus on making connections explicit on a meta-level.

A number of studies reported on learners’ difficulties understanding the hierarchical classification of quadrilaterals, and their reasoning about these being influenced by their inadequate concept images (Fujita & Jones, 2007; Heinze, 2002). Assessing prospective primary school mathematics knowledge of different geometric concepts (from school geometry) before attending the course, could be an additional measure. Consequently, becoming more sensitive to their knowledge from school geometry would allow tailoring the course content to their needs. Similarly, weekly self-assessment quizzes after each lecture could be made available on the university’s learning platform. On the one hand, such measure would allow prospective teachers with the opportunity to receive timely feedback on their individual development, and to reflect on their own progress. On the other hand, it would allow lecturers to intervene in cases where gaps or difficulties in professional knowledge are present.

6 Conclusion

The professionalization of prospective mathematics teachers is an essential aspect of mathematics teacher education programs. Though the study findings revealed a wide range of competence levels in all facets of study participants' professional knowledge, rather worrying deficits in their CK and PCK were exhibited. Prospective primary school mathematics teachers especially struggled to establish links between specific sub-tasks and the subject-specific CK and PCK conveyed in the 'Geometry and Teaching Geometry' course. These connections need to be made more explicit in mathematics education courses in order to support the development of prospective teachers' professional knowledge (Reitz-Koncebovski et al., 2022). It is also crucial to provide prospective teachers with effective concept formation models (Fujita & Jones, 2007; Moore, 1994; Vinner, 1991; Weigand, 2015) in order to improve current situation of geometry teaching at both school and university level. However, it remains open to what extent mathematics teacher education programs can fulfill this objective or even should, taking into consideration that professionalization of teachers does not end with completing a teacher preparation program. Rather teacher professional knowledge needs to be continuously developed through professional development which would allow them to acquire profound mathematical knowledge for the 'school of future'.

Research ethics

Artificial intelligence

While drafting the manuscript, the Grammarly add-on in MS Word was used to check and correct grammatical errors. DeepL's AI writing assistant was used for stylistic purposes.

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Institutional review board statement

A study-specific approval by the ethics committee of the University of Potsdam was not needed as the study involved non vulnerable adults. The study involved anonymized prospective primary school mathematics teachers' written responses on the knowledge test task, with no sensitive personal data or interventions. All original research procedures, including research with human subjects, were in accordance with the principles of the research ethics published by the American Psychological Association (2017) as well as with the ethical guidance of Article 13 of the EU General Data Protection Regulation (GDPR) (2025) which are both approved and followed by the author's institution ethics committee.

Informed consent statement

The informed consent was waived due to knowledge test (as a part of the final exam) being an obligatory part of the teacher education program.

Data availability statement

Data supporting the findings and conclusions are available upon request from the corresponding author.

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Conflicts of Interest

The author declares no conflict of interest.

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