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Constructing geometric knowledge: A narrative on circle tangencies in a generalised arbelos

Christer Bergsten

Linköping University, Sweden

Abstract: In the context of explorative mathematical work within a Dynamic Geometry Environment (DGE), this article tries to communicate the author´s construction of (established and new) geometric knowledge linked to a generalisation of the classical configuration of an arbelos. With the aim to trace the genesis of knowledge construction, the story of a discovery learning process is presented as a mathematical narrative with an educational gaze, which brings personal meaning making and institutionalised mathematical knowledge together. The story is theoretically framed by reflections on educational research on DGEs, discovery learning, and mathematical narratives. Some historical remarks on a generalised arbelos, relevant for the discussion, are also presented. Finally, some critical issues related to the narrated experience of knowledge construction and their relevance to mathematics education are discussed.

Keywords: arbelos, discovery learning, dynamic geometry environment, geometry, figural concepts

Correspondence: christer.bergsten@liu.se

1 Introduction

I thought I had made a mathematical discovery and it felt great. But then I was not really sure about its originality. Then, again, I thought it did not really matter as it did not affect the experience of discovery, a kind of satisfaction that my attempt had been successful: I had constructed a piece of geometric knowledge. And I wanted to share it with others. Not for showing how clever I had been but just because it was somehow nice. In fact, I did not feel clever at all, it just happened to work, as by chance but not completely by chance. To share it, it would be easy to write it down, mathematically. However, I did not only want to communicate the "result" but also the process and experience of discovery, which felt more important, even though process, experience and result were strongly linked. Such type of (minor or major) discoveries are experienced every day in mathematics classrooms all over the world – getting a task right, solving a problem, seeing a connection, getting an idea. Knowledge is constantly being constructed and reconstructed but is mostly not recognised as such. This paper is an attempt to communicate such an experience of knowledge construction in (school level) mathematics with an educational gaze. It took place within a context of classical geometry, employing a dynamic geometry software as supporting tool.

A theoretical framing to my communication is outlined in Section 2. Then, in Section 3, the story of my knowledge construction process is being told. Before the concluding discussion in Section 5, Section 4 offers some historical comments regarding the constructed knowledge.

2 Theoretical framing

This section has three parts: first, some critical aspects of dynamic geometry environments and discovery learning are discussed; secondly, the notion of mathematical narratives is highlighted; and third, the mathematical background for the narrative is sketched.

2.1 Dynamic geometry environments and discovery

In educational studies of students' use of dynamic geometry environments (DGE), the step from experimenting with and constructing specific geometric configurations to engaging in formal proofs for the observed figural properties, has often been a focus of concern (e.g., Mariotti, 2000, Sinclair & Robutti, 2013). The critical issue of the students' "transition from empirical to deductive reasoning" (Lachmy & Koichu, 2014, p. 151) also involves the intricate relation between the software and the theoretical structure of Euclidean Geometry (Goldenberg & Cuoco, 1998). So, Mariotti (2000) suggested that after "a construction problem is solved, i.e. if the Cabri-figure passes the dragging test, a theorem can be proved with a geometric proof" (p. 28).[1] The rationale behind the didactic interest in explorative work with a DGE suggests that "passing the dragging test" may not only invite to a proving activity but can also signal that some geometric relationships have been "discovered", which did not belong to the pre-knowledge of the individual(s) involved in the activity, or even to the mathematical community. And, in education, "discovery learning" is indeed a basic tenet in pedagogies drawing on constructivism: "The virtues of encouraging discovery are of two kinds. In the first place, the child will make what he learns his own, will fit his discovery into the interior world of culture that he creates for himself. Equally important, discovery and the sense of confidence it provides is the proper reward for learning." (Bruner, 1962/1979, p. 126, cited in Takaya, 2008, p. 7).[2]

[1] Arzarello et al. (2002) provides the following definition of *dragging test*: "moving dragable or semi-dragable points in order to see whether the drawing keeps the initial properties. If so, then the figure passes the test; if not, the drawing was not constructed according to the geometric properties you wanted it to have." (p. 67)

[2] Cf. Freudenthal's notion of *re-invention*: "The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student. It is believed that knowledge and ability acquired by re-invention are better understood and more easily preserved than if acquired in a less active way." (Freudenthal, 1973, p. 118)

In geometry, it is common to make a distinction between drawing and figure, where "Drawing refers to the material entity while figure refers to a theoretical object" (Laborde, 1993, p. 49).[3] In a DGE this distinction (and the con-temporal inseparable connection) is inherent in the menu icons, as the "drawing" that occurs on the computer screen after selecting, for example, the icon for "midpoint perpendicular" in the menu and clicking on two points on the screen, is not just a line but a line that owns the selected property. In drag mode one can move this line only by dragging objects previously used for its construction; its selected property thereby remains preserved. The DGE thus, it seems, supports what Fischbein (1993, p. 154) calls "the total fusion between the conceptual and the figural aspects" in geometric thought. To "pass the dragging test" in a DGE is then for an observed invariance a strong indication of the validity of an underlying geometric property. Therefore, in "discovery learning", within a didactical design context or in self-initiated mathematical work within a DGE, the following description is deeply relevant: "usually in the process of mathematical invention we try, we experiment, we resort to analogies and inductive processes by manipulating not crude images or pure, formal axiomatic constraints, but figural concepts, images intrinsically controlled by concepts" (Fischbein, 1993, p. 160; italics in original). Leung (2015), discussing potentials of different types of dragging in a DGE, draws on Fishbein's notion of figural concepts in seeing a DGE as "a means to access Euclidean Geometry through the dragmode" (p. 467).

While the term discovery learning covers a wide range of instruction practices, the meta-analysis across school subjects by Alfieri et al. (2011) suggested that "unassisted discovery does not benefit learners" (p. 1) but that guided forms of discovery instruction do. In mathematics education, the term *mathematical discovery processes* has been used to denote types of activities that take place when students engage in discovery learning (Leuders & Philipp, 2013; see also Danzer, 2024). After a concise research summary, Amaral-Schio and Haug (2020) listed a number of advantages in employing a DGE in the geometry classroom, within main approaches *exploration*, *proof* and *visualization*. While usage differed, the goal remained the same: "to provide an environment for the student to construct knowledge of the field of geometry" (p. 1).

2.2 Mathematical narratives

In this paper the construction of geometric knowledge will be discussed in both a literal and a theoretical sense; literal, while remaining within geometric constructions in the classical Euclidean mode, employing a DGE as the main construction tool; and theoretical, while keeping the main interest on knowledge growth. The mathematical topic used for the discussion, which will be run in a conglomerate of a narrative and formal mathematics, involves the construction of tangent circles related to a generalization of the classical

[3] This distinction can be traced back to Fischbein´s notion of "figural concepts" (e.g., Mariotti & Fischbein, 1997).

configuration of an arbelos.[4] By employing a narrative form, rather than presenting a final and nicely polished mathematical end product, the aim is to genetically trace the very construction of geometric knowledge; that is, how observations, constructions, generalisations, and formulas come to life in a sequence of goal-directed actions. This aim thus deeply involve processes of learning and knowing mathematics.

Drawing on the work of Bruner (e.g., 1986), Mor and Noss (2008) define narrative "as a progression of statements describing something happening to someone in some circumstances" (p. 216), key elements of which are context, plot and purpose (p. 217). Similarly, Healy and Sinclair (2007) describe a narrative mode as "a discursive mode in which /…/ past experiences are being recapitulated in a sequence of clauses" (p. 7). For their conceptualisation of a mathematical narrative[5], referring to Bruner's (1986) distinction of narrative and paradigmatic modes, they list four "characteristics" attached to it: inherent sequentiality, about real or imaginary events, connecting the exceptional and the ordinary, and "some kind of dramatic quality" (p. 8). For the last of these, they provide the example of how the story told by Hofstadter (1997) about his discovery of a "geometric gem" influenced his mathematical thinking. In the discovery process, his use of a DGE played an active role, "filling an inner need, a craving, that I had, to be able to see my beloved special points doing their intricate, complex dances inside and outside the triangle as it changed" (Hofstadter, 1997, p. 13). In my reading of his "story", narrative and paradigmatic discourses complemented each other, thus connecting personal meaning making and institutionalised mathematical knowledge. Healy and Sinclair (2007, p. 20) argue that "expressive technologies" (such as DGE) can support narratives to become productive in this sense, not only for mathematicians but also for students learning mathematics. In this context they use the term *ideational mathematics* for "the ideas of mathematics actual people have when they are working on mathematical problems or discussing among peers" (p. 7), which can be expressed through narratives. In this vein, the narrative in Section 3 takes an educational gaze on how the DGE-supported knowledge construction process brings the author's ideational knowledge and the institutionalised knowledge together.

2.3 Mathematical focus

I once spent some time "experimenting" with an *arbelos* in a DGE, focussing on constructions and properties of the incircle, the Archimedean twin circles, and circle chains.[6] So, I

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[4] Mathematically, it will be assumed that the reader is familiar with basic notions and constructions in classical (Euclidean) geometry, including *power line* (also called *radical axis*), *centre of similitude (*or *similarity*; also called *homothetic centre)*, and *circle inversion*. The mathematical requirements will not go beyond those at school level.
[5] Healy and Sinclair (2007, pp. 6-8) refer here to Bruner's (1986) distinction of narrative and paradigmatic modes, as well as their own constructs ideational and paradigmatic mathematics.
[6] Definitions of this classical configuration, including the twin circles and the incircle, can be found for example in the oldest known source Archimedes (n.d./2018). In Figure 1, the arbelos is the area between the (semi)circles with diameters BC (outer circle), BF and FC (inner circles), having their centres at A, D and E, respectively. FG is (part

drew some circles passing, as it looked like, key points of the arbelos. It soon became clear that three pairs of circles were strongly (or magically, if you like) linked to these classical tangency problems. It was thus shown in Bergsten (2006) how two pairs of *outer magic circles* with centres at the endpoints B and C of the diameter, and passing F and G, respectively, are linked to the twin circles, as illustrated in Figure 1 (left):

- the centre points of the twin circles lie on the power lines to these pairs of outer magic circles, passing the intersection points R and R', respectively,
- the outer magic circles by G intersect the given inner circles of the arbelos at the tangency points of the twin circles
- the non-common vertical tangents of the twin circles intersect the given inner circles at the tangency points of the common tangent to the given inner circles of the arbelos, and pass the intersection points P and P' of the (outer) magic circles by F and the given outer circle of the arbelos.

Figure 1 (right) shows the third pair of such (inner) "magic" circles, C_{HB} and $C_{H'C}$,[7] having their centres on BC and constructed to pass the intersection points R and R' of the two pairs of magic circles just described, as well as B and C, respectively[8]. In Bergsten (2006) it was shown that (cf., Figure 1, right):

- the centre M of the arbelos incircle lies on the power line to the (inner) magic circles C_{HB} and $C_{H'C}$
- the (inner) magic circles intersect the given inner circles of the arbelos at the tangency points of the incircle
- the diameters of C_{HB} and $C_{H'C}$ are the harmonic means of the diameters of C_{AB} and C_{DB}, and of C_{AB} and C_{EC}, respectively

In addition, it was demonstrated how three circle chains in the arbelos are intrinsically linked to these three pairs of magic circles (Bergsten, 2006, Figure 16).[9]

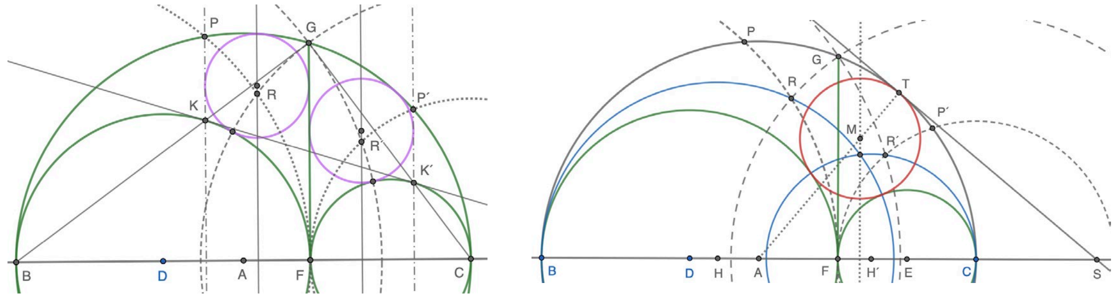
of) the powerline to the circles with diameters BF and FC. The incircle is tangent to all three semicircles, while the twin circles are tangent to the outer circle, one inner circle and the power line.

[7] Throughout this paper, the notation C_{MP} will refer to a circle with centre at point M and radius MP, where P is a point on the circle.

[8] H and H' can be found as the intersection points between BC and the midpoint perpendiculars to B and R, and C and R', respectively.

[9] These circle chains to the arbelos incircle were also (independently) discussed in Danneels and van Lamoen (2007); what they called *midcircles* (Dixon, 1991; see also Coxeter and Greitzer, 1967, p. 121), defined to map two given circles on each other by circle inversion, coincide in an arbelos with the magic circles as defined above.

Figure 1. Magic circles in an arbelos and connections to twin circles (left) and incircle (right)



Note. The left image shows an arbelos with two pairs of (outer) magic circles (dotted) and the Archimedean twin circles. FG is the power line to the given inner circles. The right image shows the third pair of magic circles and the incircle. S is the (external) centre of similitude to the circles C_{DB} and C_{CE} and ST is the tangent from S to the circle C_{AB}.

In an arbelos, the many "marvelous" properties (cf., Bankoff, 1994) draw on a very special case of a power line to two given circles that are tangent to each other, with the power line passing the tangency point, which I used as basis for the construction of the magic circles. I was curious to explore these properties for the more general case with the inner circles not being tangent, their power line being situated between them or intersecting them. So, I went on to further investigate the role of the power line and the magic circles for these tangency problems by 'opening up' the arbelos at the tangency point F in Figure 1. This generalisation of an arbelos, which I called an *open arbelos* (Bergsten, 2009, p. 22), is illustrated in Figure 2.[10] When exploring this configuration using a DGE, I could observe that many of the properties listed above indeed still passed the dragging test, while some did not. Interesting from an educational perspective was how these observations strongly invited to the production of "explanations" and/or proofs, as well as further explorative work, which will be narrated in the following section. An overview of the somewhat disordered geometric knowledge being constructed throughout the narrative will follow in Section 5.

3 A mathematical narrative

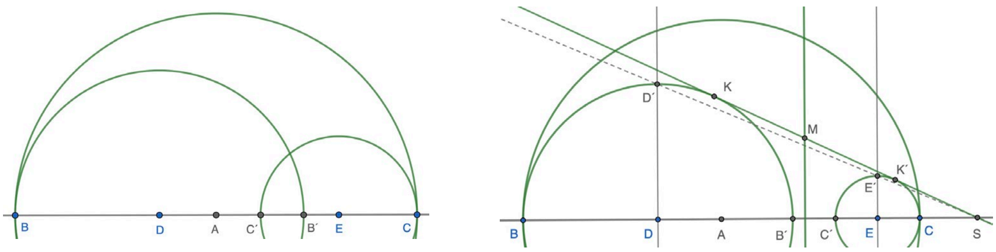
The first part of this section has its focus on the notions of an open arbelos, power line, and magic circles. In the second part geometric knowledge involving the incircle is constructed, followed by some generalisations. The twin circles are re-investigated and further explored in the third part, followed in part four by some observations of connections between incircles and twin circles.

[10] Other kinds of generalisations of the arbelos have been investigated, as for example exchanging the circle to other conics such as the parabola and the ellipse (Różański et al., 2017), or using 3D or higher dimension configurations (Oller-Marcén, 2016); see also Okumura & Watanabe (2008; 2009). The simple (and "natural") generalisation I call an *open arbelos* lends itself to investigations drawing on basic Euclidean geometry at school level.

3.1. The power line and magic circles in an open arbelos

It was a kind of adventure to explore circle tangencies in this "new" territory of an open arbelos, as at the time I had, to my surprise, not come across any writings about it; of course, I did not really systematically search for it. My limited initial explorations on this geometric configuration were reported in Bergsten (2009), with focus on the twin circles, drawing on geometric constructions and algebraic elaborations (see Section 3.3). Figure 2 shows an open arbelos, with intersecting (left) and disjoint (right) inner (semi)circles, with centre points on the diameter of the outer (semi)circle, keeping the tangencies to the outer circle. An arbelos is then a special case of an open arbelos (when the points B' and C' in Figure 2 coincide). The power line to the inner circles must pass the midpoint M between the tangency points of the common tangent (Figure 2, right).

Figure 2. Two configurations of an open arbelos

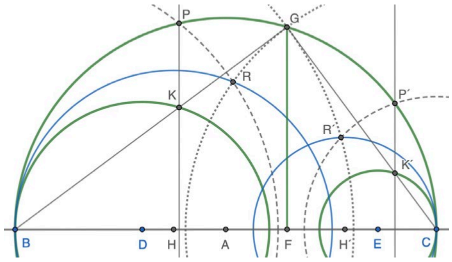


Note. The righthand image also shows a construction (by the dotted line) of the external centre of similitude S (here, the radii DD' and EE' are perpendicular to BC and thus parallel), and a common tangent (by S, K', and K), along with the power line to the inner circles by M. See further Appendix E.

Exploring Figure 3, where FG is (a segment of) the power line to the inner circles C_{DB} and C_{EC} in an open arbelos, a DGE dragging activity suggested that also for an open arbelos the segments BG and CG intersect these circles at the tangency points K and K' of the common tangent, respectively. This of course also needed a proof (see Bergsten, 2009).

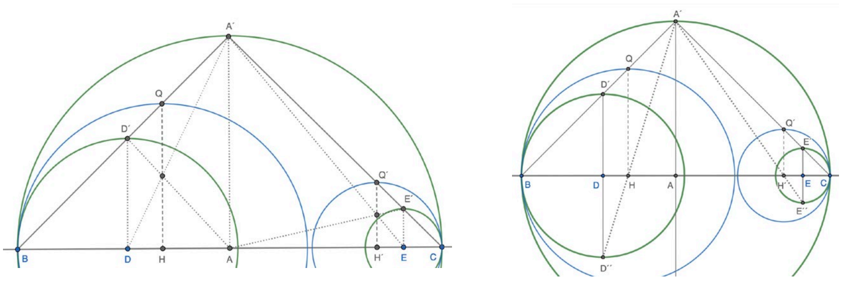
Now, to parallel the procedure I had used for the construction of the inner magic circles in an arbelos (Section 2.3), I defined R as the intersection point of the outer circles C_{BP} and C_{CG}, and R' as the intersection point of the other pair of outer circles, $C_{CP'}$ and C_{BG} (cf., Figure 1). The circles C_{HB} and $C_{H'C}$ could then be drawn by letting them pass B and R, and C and R', respectively, having their centre points H and H' on the diameter BC (see Figure 3, where H and H' were found by drawing the perpendicular bisectors to the segments BR and CR').

Figure 3. The power line (FG) and the three pairs of (magic?) circles in an open arbelos



However, using the drag mode I could notice that with a small radius DB and/or EC, the intersection points R and R' may not exist! Could this problem really be solved? I tried to manage this unexpected behaviour by constructing the circles C_{HB} and $C_{H'C}$ based on the harmonic mean property discussed in relation to Figure 1. The left image in Figure 4 shows how I did this, drawing on a standard construction of the harmonic mean HQ of the radii AA' and DD', and H'Q' of the radii AA' and EE' (e.g., Yiu, 2005, p. 82). I could then indeed confirm by dragging that the two constructions produce the same magic circles when the intersection points R and R' exist. The construction being somewhat awkward, however, I asked myself if this perhaps could be done more smoothly. After some reflection, I realised that a more direct way was to use the diameters rather than the radii for the same type of construction, as in the righthand image in Figure 4 (where H is the intersection of BC and A'D', and H' the intersection of BC and A'E'). It then also became visible, as a bonus, that the intersection point H is the internal centre of similitude to the circles C_{AB} and C_{DB} (and similar for H').[11] I then felt comfortable to *define* the inner magic circles C_{HB} and $C_{H'C}$ by the harmonic mean construction in Figure 4. And I could experience these explorations and reflected observations as a genuine learning process.

Figure 4. Constructions of the inner magic circles



In search for a construction of the incircle, which for an arbelos had its centre on the power line to the inner magic circles, I now wanted to situate this power line in an open arbelos algebraically. In addition to the Euclidean discourse, algebraic representations and operations could here support the construction of geometric knowledge – not only for

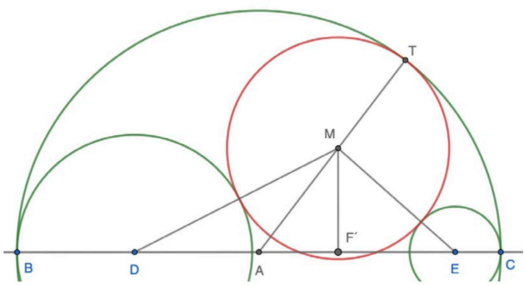
[11] In line with Danneels and Lamoen (2007), this means that the inner magic circles, as constructed in Figure 4, also for an open arbelos are midcircles to C_{AB} and C_{DB} / C_{EC}.

verification but also as an aid for the actual construction.[12] Thus, as in Bergsten (2009), taking BC and its extension as a coordinate axis with B as origin and BC as length unit, I used the following notations (cf. Figure 2, Figure 3): $BC = 1, BB' = r_1, BC' = r_2, BF = a$. To keep the left inner circle bigger than the right one (symmetry argument), I here also assumed that $r_1 > 1 - r_2$ (i.e., $r_1 + r_2 > 1$). Then, drawing on similar triangles, the x-coordinate[13] of S was easily found to be $\frac{r_1 r_2}{r_2-(1-r_1)}$, and then $a = \frac{r_2}{r_2+1-r_1}$. With this value of a (i.e., the x-coordinate of F in Figure 3, situating the power line to the circles C_{DB} and C_{EC} on BC), the x-coordinates of K and K' were found to be ar_1 and $a + r_2(1-a)$, respectively. The difference between these x-coordinates was calculated to be $\frac{2(1-r_1)r_2}{1-r_1+r_2}$, which thus equals the harmonic mean of $1 - r_1$ and r_2. Further (with reference to Figure 4), the radius HQ was found to be $\frac{r_1}{1+r_1}$ and the radius H'Q' $\frac{1-r_2}{2-r_2}$. These values equal half the harmonic mean of the radii of C_{AB} and C_{DB}, and of the radii of C_{AB} and C_{EC}, respectively.[14] I found all these formulas not only "simple" and useful as bases for illustrative geometric constructions but also theoretically compelling as "natural" generalisations from the arbelos case.

3.2 The incircle of an open arbelos

As indicated above, looking at the arbelos case, an obvious hypothesis for the construction of the incircle in an open arbelos would be that its centre is on the power line to the inner magic circles. So, to find the x-coordinate of the centre point of the incircle, I performed an algebraic analysis based on a drawing of an incircle (Figure 5).[15]

Figure 5. The incircle of an open arbelos



Note. F' is the intersection point of the perpendicular to BC from the incircle centre point M, and T is the tangency point of the incircle to the given outer circle C_{AB}.

[12] For a discussion of this issue and some examples, see Yiu (2005).
[13] Here, the x-coordinate in (x, y) refers to the position on BC (as in Section 3.1) and the y-coordinate to the perpendicular distance from BC.
[14] This "mirrors" the arbelos case, where $r_1 = r_2 = r$, and the correspoonding radii are $\frac{r}{1+r}$ and $\frac{1-r}{2-r}$ (Bergsten, 2006). Also the open arbelos formulas for S and a above mirror the arbelos case in the same way.
[15] In Figure 5, the open arbelos has disjoint inner semicircles. However, all properties and formulas derived in this section indeed apply also to an open arbelos with intersecting inner semi-circles (the value a in Section 3.1 and the equations in (1) remain the same).

Using the notation introduced in Section 3.1, the coordinates (α, β) of the centre point M of the incircle, and its radius ρ, were sought by solving the system of equations (1) below, based on the application of Pythagoras' theorem on each of the triangles MF'E, MF'D, and MF'A in Figure 5. The unique solution, which was easily found, is also presented:

$$\begin{cases} \left(\alpha - \frac{1+r_2}{2}\right)^2 + \beta^2 = \left(\rho + \frac{1-r_2}{2}\right)^2 \\ \left(\alpha - \frac{r_1}{2}\right)^2 + \beta^2 = \left(\rho + \frac{r_1}{2}\right)^2 \\ \left(\alpha - \frac{1}{2}\right)^2 + \beta^2 = \left(\frac{1}{2} - \rho\right)^2 \end{cases} \tag{1}$$

$$\begin{cases} \rho = \frac{(1-r_1)r_2}{2(1-r_1+r_1r_2)} \\ \alpha = \frac{(1+r_1)r_2}{2(1-r_1+r_1r_2)} \\ \beta = \frac{\sqrt{r_1r_2(1-r_1)(1-r_2)}}{1-r_1+r_1r_2} \end{cases} \tag{2}$$

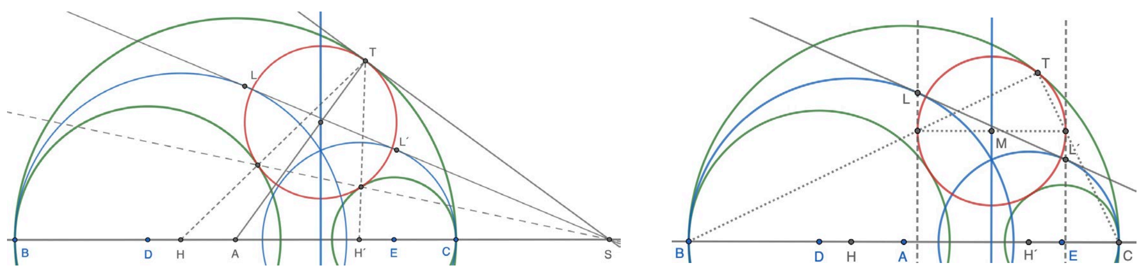
The formula in (2) for the radius ρ of an open arbelos incircle thus turned out to be a "direct" generalisation of the corresponding formula $\frac{(1-r)r}{2(1-r+r^2)}$ for the case of an arbelos! The experience of discovering such an algebraic correspondence between an arbelos and an open arbelos, strongly triggered further exploration. And indeed, also the formula for α mirrors the corresponding formula $\frac{(1+r)r}{2(1-r+r^2)}$ for the arbelos (cf. Bergsten, 2006). Moreover, a straightforward calculation showed that the position on BC of the power line to the magic circles C_{HB} and $C_{H'C}$ (Figure 6) is $\frac{r_2(1+r_1)}{2(1-r_1+r_1r_2)}$, which thus coincides with the value of α in (2)!

Furthermore, a direct calculation showed that the incircle diameter (i.e., 2ρ) equals the harmonic mean of r_2' and $1-r_1'$, where r_1' and r_2' are the x-coordinates of the intersection of BC and the magic circles C_{HB} and $C_{H'C}$, respectively. As the tangency point T of the incircle to the circle C_{AB} is where the tangent to C_{AB} from the centre of similitude S to the inner circles touches C_{AB} (see Appendix A), the constructed "new" knowledge could be formulated as the following proposition:

Proposition 1
The centre of an open arbelos incircle is the intersection point of the power line to the (inner) magic circles and the radius of the outer circle to the tangency point of its tangent from the centre of similitude to the inner circles.

The situation is shown in Figure 6. Not only was the hypothesis above confirmed, Proposition 1 also presents one way to construct an open arbelos incircle. As discussed above for the inner circles, applied to the magic circles as given inner circles of an open arbelos, the difference between the x-coordinates of L’ and L must be the harmonic mean of $r_2{}'$ and $1 - r_1{}'$, which thus equals the diameter of the incircle. Indeed, the vertical tangents to the incircle do pass L and L’, respectively[16], and, in line with the lemma in Appendix B, touch the incircle by the intersection points of the incircle and the segments BT and CT, respectively (Figure 6, righthand image).

Figure 6. The incircle of an open arbelos



I could also observe (Figure 6), and confirm by dragging, that the segments HT and H’T meet the given inner circles C_{DB} and C_{EC} at their tangency points to the incircle C_{MT}![17] Interestingly, the line by these tangency points passes S. These observations were then for me unexpected and non-trivial, thus inviting to an activity of explanation. Now, since C_{HB} by circle inversion maps the circle C_{AB} on C_{DB}, and the incircle on itself[18], the tangency point T of the incircle to C_{AB} must map on C_{DB} at the tangency point of the incircle – and similar for the tangency of the incircle to C_{EC} and the inversion in $C_{H'C}$. These mappings could easily be performed in a DGE using the circle inversion command. As by circle inversion the circle C_{ST} maps the circles C_{DB} and C_{EC} on each other, also these tangency points are mapped on each other, and the line connecting them must pass the centre of the inversion circle C_{ST}. I would call this argument an explanation, based on the justification for why the circle C_{ST} maps the circles C_{DB} and C_{EC} on each other (and the incircle C_{MT} on itself). So, the DGE applications and the argumentation seem to work here directly by way of figural concepts. The situation is shown in Figure 7, where the (arbitrary) point P on C_{DB} is mapped on Q on C_{EC} by inversion in C_{ST}. Selecting P = B’, the equation $(s - r_1)(s - r_2) = |ST|^2$ should then hold, where s is the x-coordinate of the point S (as in Section 3). A direct but rather boring calculation could verify this equality.

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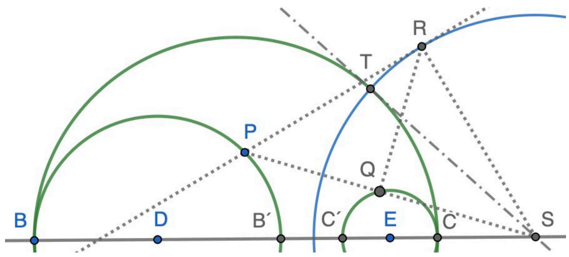
[16] Drawing on the formulas for S, a, and K in Section 3.1, the x-coordinate of the perpendicular to BC by L was found to be $\frac{r_1 r_2}{1 - r_1 + r_1 r_2}$, which indeed equals the difference between the x-coordinate of the incircle centre and its radius (i.e., $\alpha - \rho$).

[17] As pointed out in section 2.3, these tangency points coincide for an arbelos with the intersection points between the inner magic circles and the given inner circles, a property thus not generalisable to an open arbelos.

[18] As Danneels and van Lamoen (2007, p. 57) pointed out for the case of an arbelos, the incircle in an open arbelos also maps on itself by inversion in C_{HB} and in $C_{H'C}$.

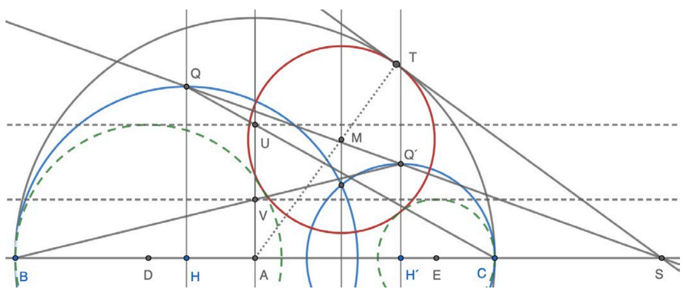
Figure 7. Inversion in the circle C_{ST} (angles PRS and PQR are 90°)



Working within a DGE sometimes leads to an unexpected observation that can be used for a new (and thus also unexpected) construction. For example, the observation above that the segments HT and H'T meet the given inner circles C_{DB} and C_{EC} at their tangency points to the incircle C_{MT} (Figure 6) could be used for a construction of the incircle that does not draw on the power line to the magic circles C_{HB} and $C_{H'C}$. Indeed, by drawing the line l by D parallel to AT, the segment from T to the intersection point below BC of l and C_{DB} passes H, and meets C_{DB} at its tangency point to the incircle (Appendix B). Finally, (for example) the intersection of AT and the midpoint perpendicular to this tangency point and T produces the centre of the incircle.

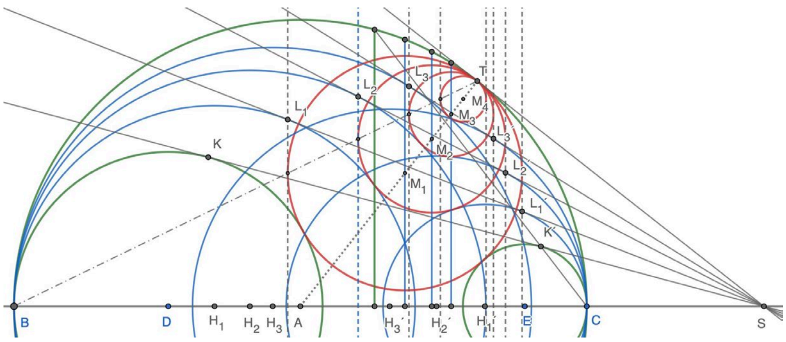
While the magic circles C_{HB} and $C_{H'C}$ by circle inversion map the outer circle C_{AB} on the inner circles C_{DB} and C_{EC}, respectively, I also "discovered" that it is quite easy to re-*construct* the inner circles (or, to 'find' them), given their magic circles, by drawing the segments BQ' and CQ, as indicated in Figure 8. The length x of AU then equals the radius of the inner circle C_{DB} (and AV of C_{EC}). Indeed, for the left-hand circle, with h denoting the length of HQ = HB and R the length of AB, x can be found from the equation $\frac{h}{2R-h} = \frac{x}{R}$ (triangles CHQ and CAU are similar), which can be written $h = \frac{2Rx}{R+x}$. This means that h is the harmonic mean of x and R, showing that C_{HB} is the (left) inner magic circle to C_{AB} and C_{DB}. The tangency point of the (dotted) horizontal line by U and the circle with radius x on BC by B is found as the intersection of BQ and the horizontal line by U. Thus, given an open arbelos, (inner) magic circles can be constructed both "outwards" and "inwards".

Figure 8. A construction of magic circles "inwards"



In line with Fischbein´s comment about "the process of mathematical invention" (as cited in Section 2.1), I could further explore the configuration in Figure 6. Since Proposition 1 holds for any choice of r_1 and r_2 with $0 < r_1 < 1$, $0 < r_2 < 1$, $r_1 + r_2 > 1$, it will apply also to the magic circles C_{HB} and $C_{H'C}$, seen as the inner circles in C_{AB} of an open arbelos; that is, for a construction of their incircle, their (inner) magic circles can be constructed. This way a chain of pairs of magic circles, and their incircles, can be constructed, drawing on the result of Proposition 1. So, starting with given inner circles C_{DB} and C_{EC} and outer circle C_{AB} in an open arbelos, to produce Figure 9, successive pairs of inner magic circles were iteratively constructed, along with their common tangents and incircles (C_{M_kT}). It thus makes sense to talk about first order magic circles (C_{H_1B} and $C_{H_1'C}$ in Figure 9), whose magic circles (C_{H_2B} and $C_{H_2'C}$) are defined as second order magic circles, and so on, along with the corresponding incircles. By Proposition 1, the second order incircle (i.e., C_{M_2T} in Figure 9) must have its centre point on the power line to the second order magic circles. However, that the notion of the order of (inner) magic circles has a relevance for tangency problems in an open arbelos, became clear first after further work with the DGE, and could finally be formulated as Proposition 2 (see Section 6).

Figure 9. A chain of incircles in an open arbelos



Note. The figure shows common tangents and incircles (C_{M_kT}) to pairs (C_{H_kB} and $C_{H_k'C}$) of a chain of inner magic circles in an open arbelos. The vertical dotted lines are tangents to the incircles.

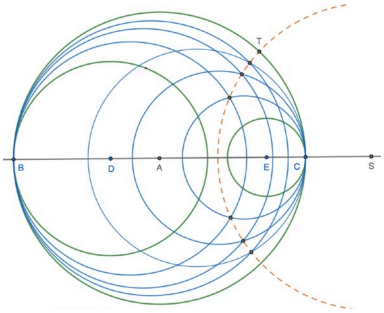
In Figure 9 it can also be observed that the common tangents to pairs of the magic circle chain all pass the centre of similitude S to the two given inner circles of the open arbelos (cf. Appendix E). To verify this, one only needed to calculate the centre of similitude to the first order magic circles C_{HB} and $C_{H'C}$. Additionally, as noted above (and seen in Figure 9), the lines perpendicular to the diameter BC by the tangency points L_1 and $L_{1'}$, respectively, are tangent to the incircle C_{M_1T}. That this type of tangency holds for all pairs of magic circles in the chain in Figure 9 then follows inductively but can also be “explained” by the Lemma in Appendix B, while noticing that the segments BT (indicated in Figure 9) and CT pass all tangency points of the incircles to these vertical tangents.

Using the DGE, I could also observe that the incircles in Figure 9 are themselves “magic”: by circle inversion in C_{M_kT} the outer circle C_{AB} is mapped on the “next” incircle $C_{M_{k+1}T}$. An open arbelos is really full of magic circles!

Figure 10 shows how the circle C_{ST} passes all the intersection points between pairs in the inner magic circle chain. This can be “explained” by applying circle inversion to such a pair of circles with respect to the circle C_{ST}: the intersection points with the circle C_{ST}

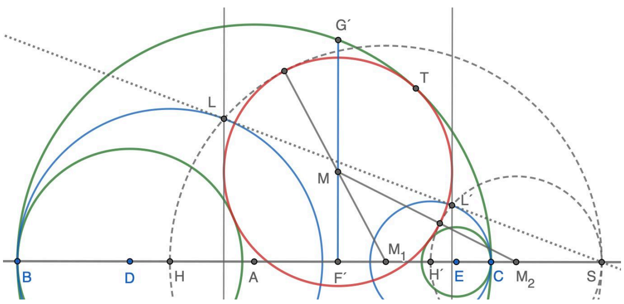
remain fixed and must thus be identical since such a pair of circles map on each other[19]. C_{ST} also maps C_{AB} and the incircles on themselves.

Figure 10. Intersection points of pairs of circles in a magic circle chain lie on the circle C_{ST}



Another unexpected "empirical discovery" was made during the work on the incircle. Using drag mode, it became clear that the circles with diameters HS and H'S, respectively[20], are tangent to the incircle C_{MT} (Figure 11)! This observation could then be verified algebraically by the application of Pythagoras' Theorem, drawing on the parameters α, β and ρ above, and the coordinates for the points H, H' and S (see Appendix C). A more interesting "explanation" can be provided by circle inversion: the incircle is by inversion in C_{ST} mapped on itself and its vertical tangents are mapped on these two circles (i.e., circle C_{M_1S} on the tangent by L' and circle C_{M_2S} on the tangent by L; Figure 11)[21], or vice versa for the purpose of discovery. This type of tangency certainly holds for any order of magic circles / incircles.

Figure 11. Circles tangent to the open arbelos incircle



Note. The two circles with diameters HS and H´S and centres M_1 and M_2 on the segment BS are tangent to the open arbelos incircle.

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[19] That is, the circle C_{ST} is a midcircle to the midcircles (the inner magic circles).
[20] These are the circles used for finding the tangency points (L and L´) for the (common) tangent from S to the circles C_{HB} and $C_{H'C}$.
[21] As the circles with diameters HS and H'S pass S, which by inversion in C_{ST} maps into infinity, they are mapped on lines l_1 and l_2 perpendicular to BC (as these circles are perpendicular to BC at H and H'). As the common tangent to C_{HB} and $C_{H'C}$ (by L and L') maps on itself and C_{HB} and $C_{H'C}$ on each other, L and L' map on each other. Therefore, as the circles with diameters HS and H'S pass L and L', l_1 and l_2 must pass L and L'. So, the circles with diameters HS and H'S are by inversion in C_{ST} mapped on the vertical tangents to the incircle.

3.2.1 Interlude: Diameters of inner magic circles and incircles of different orders

The newly constructed knowledge in Proposition 1 and its generalisation in Figure 9 was established thanks to an initial algebraic analysis, as well as the use of figural concepts. Not really knowing where it would lead, I could not resist taking on a further analysis of this generalisation algebraically, which will be reported here.

Thus, for the n^{th} pair of (harmonically) constructed inner magic circles, with circles $\mathrm{C}_{\mathrm{H}_1\mathrm{B}}$ and $\mathrm{C}_{\mathrm{H}_1{}'\mathrm{C}}$ representing $n = 1$ (Figure 9), the endpoints of their diameters on BC (≠ B and ≠ C) were denoted by B_n for the left and C_n for the right circle chain. I then let b_n denote to the length of the segment BB_n and c_n the length of the segment BC_n. After some algebraic work, the following formulas could eventually be found (with $n \geq 1$):

$$\begin{cases} b_n = \dfrac{2^n r_1}{1 + (2^n - 1)r_1} \\ c_n = \dfrac{r_2}{r_2 + 2^n(1 - r_2)} \end{cases} \tag{3}$$

These formulas[22] were established recursively from $HQ = \frac{r_1}{1+r_1}$ and $H'Q' = \frac{1-r_2}{2-r_2}$ (Section 3.1), directly (with $\mathrm{BB}_1 = 2\mathrm{HQ}$) producing $b_1 = \frac{2r_1}{1+r_1}$ and (with $BC_1 = 1 - 2H'Q'$) $c_1 = \frac{r_2}{2-r_2}$. For b_n, a proof by induction was accomplished by substituting $b_n = \frac{2^n r_1}{1+(2^n-1)r_1}$ for r_1 in the formula for b_1:

$$b_{n+1} = \frac{2b_n}{1 + b_n} = \frac{2 \cdot \dfrac{2^n r_1}{1 + (2^n - 1)r_1}}{1 + \dfrac{2^n r_1}{1 + (2^n - 1)r_1}} = \frac{2 \cdot 2^n r_1}{1 + 2^n r_1 - r_1 + 2^n r_1} = \frac{2^{n+1} r_1}{1 + (2^{n+1} - 1)r_1}$$

The proof is as straightforward for c_n. Using the coordinate for c_n, the diameter of the circle on $\mathrm{C}_n\mathrm{C}$ was found to be $\frac{1-r_2}{1-(1-2^{-n})r_2}$. The diameter for the circle on BB_n is b_n, which in a similar way may be written in the form $\frac{r_1}{r_1+(1-r_1)2^{-n}}$. As both these expressions are increasing sequencies smaller than 1 and approach 1 as $n \to \infty$, these circle chains (as also seen using the DGE) approach the outer circle C_{AB} from within as $n \to \infty$.

Drawing on the expressions for b_n and c_n in (3) and the incircle radius $\rho = \frac{(1-r_1)r_2}{2(1-r_1+r_1r_2)}$, the radius ρ_n of the n^{th} incircle could be expressed by the following (beautiful?) formula:[23]

$$\rho_n = \frac{r_2(1 - r_1)}{2(r_2(1 - r_1) + 2^{n-1}((1 - r_1)(1 - r_2) + r_1 r_2))} \tag{4}$$

---

[22] The diameters of the n^{th} left and right inner magic circles are then b_n and $1 - c_n = \frac{2^n(1-r_2)}{r_2+2^n(1-r_2)}$, respectively.

[23] For $n = 1$ this formula thus provides the radius of the incircle $\mathrm{C}_{\mathrm{M}_1\mathrm{T}}$ in Figure 9, for $n = 2$ of the incircle $\mathrm{C}_{\mathrm{M}_2\mathrm{T}}$, and so on. The specific form of the formula for ρ_n, to be compared to formula (2) above, was found inductively.

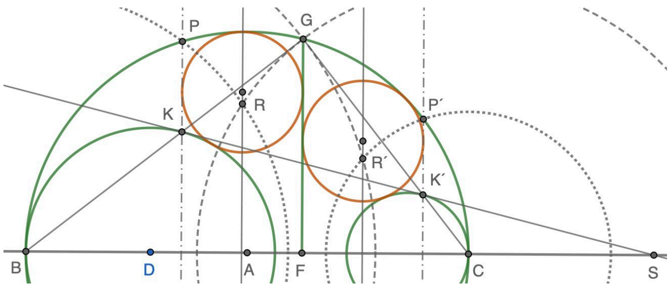
By using the distance measurement tool in a DGE, what I would call *empirical confirmation* of this formula for some beginning values of n (1, 2, 3, 4, …) did for me provide a "sense of confidence" (Section 2.1) in my process of knowledge construction, and thus has an educational value. In addition, it could even further extend this process as it indicated that this formula also holds for all $n \leq 0$. Indeed, for some different values of r_1 and r_2, I successfully confirmed the formula for $n = 0$, $n = -1$, and $n = -2$! This corresponds to what in relation to Figure 8 above was called constructing the magic circles "inwards".

After this excursion into algebraic generalisations of the incircle, it was time to revisit the twin circles.

3.3 Twin circles in an open arbelos

It was shown in Bergsten (2009) how the two circles tangent to the power line, the outer circle and one of the two inner circles in an open arbelos can be constructed (Figure 12). Furthermore, it was proved that they are congruent, that is, twin circles, as in an arbelos, each with a diameter equal to $\frac{r_2(1-r_1)}{r_2+1-r_1}$, which is half the harmonic mean of r_2 and $1 - r_1$.[24] The coordinates of the centre points of the twin circles were also determined (Bergsten, 2009, p. 31). Now, drawing on the previous discussion in this narrative, one way to "understand" this result in an open arbelos is to "read" the diameter of a twin circle as the distance between its vertical tangents (i.e., the power line FG and the perpendiculars to BC by K or K'; see Figure 12).[25] Thus, the sum of the diameters of the twin circles equals the harmonic mean of r_2 and $1 - r_1$, which equals the distance between the x-coordinates of K and K'. Somehow, this echoed the situation for the incircle as reported in Section 3.1!

Figure 12. The twin circles in an open arbelos

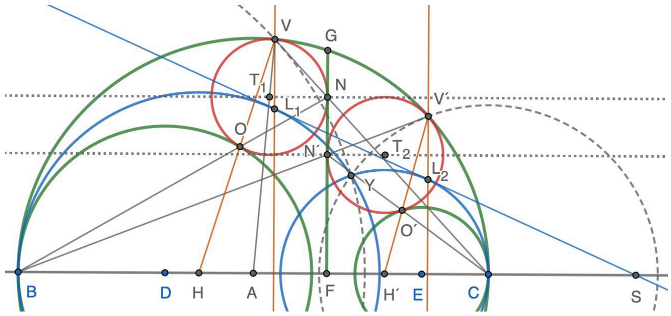


Note. The dotted lines are vertical tangents to the twin circles (cf. Figure 1 for the arbelos case).

[24] This formula can be read as a "natural" generalisation of the arbelos case, where the diameter of a twin circle is half the harmonic mean of the diameters r and $1 - r$ of its inner semicircles. See also Okumura (2012), who used the term "generalized arbelos" for what I called an open arbelos.
[25] It could easily be shown algebraically that the x-coordinate of K equals the difference between the x-coordinate of the centre point of the left twin circle and its radius (similar for the right twin).

In the arbelos case, the outer magic circles C_{BG} and C_{CG} passed the tangency points of the given inner circles and the twin circles (Figure 1). This did not happen in an open arbelos (as can be seen in Figure 12). However, it could be observed that by circle inversion in the magic circles C_{BG} and C_{CG}, respectively, the twin circles are mapped on themselves, as well as by circle inversion in the outer circles C_{BP} and $C_{CP'}$ (which thus also are “magic”; cf. Figure 1). Note that the centre points of the twin circles are on the power lines (by R and R') of the magic circles C_{BP} and C_{CG}, and $C_{CP'}$ and C_{BG}, respectively (as in the arbelos case). Now, noting that R and R' also lie on the inner magic circles C_{HB} and $C_{H'C}$ (Section 3.1), I asked myself if there was a way to (re)construct the twin circles related to the inner magic circles – how could this be done? I decided to explore the intersection point Y of the inner magic circles (Figure 13) and could then note, and confirm by dragging, that the circles C_{BY} and C_{CY} pass the outer tangency points V and V', respectively, of the twin circles! This rather surprising observation could indeed be used for a construction of the twin circles by drawing the lines BV' and CV, which must pass the tangency points N' and N, respectively (Appendix B).[26] In addition, as the segments BN and CN' must pass the tangency points O and O', respectively, three points on each twin circle could now be found.

Figure 13. Constructions of the twin circles in an open arbelos

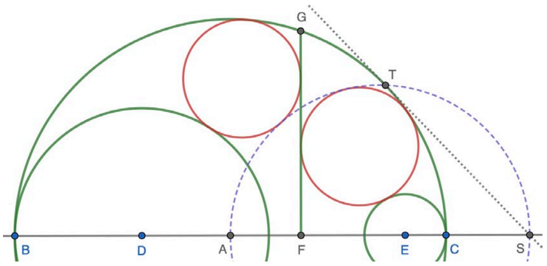


However, as the inner magic circles may not intersect (Section 3.1), further “empirical” exploration with the DGE was needed here, and eventually paid off. It turned out that also the perpendiculars to the diameter BC passing the tangency points L_1 and L_2 of the common tangent to C_{HB} and $C_{H'C}$ pass V and V', respectively (Figure 13), independently of whether the inner magic circles intersect or not! This observation could indeed be verified algebraically, drawing on the centre coordinates and radius of the incircle (Section 4) and the twin circles (Bergsten, 2009, p. 26, 28). Interestingly, letting Q be the intersection point (below BC) between C_{DB} and the line by D parallel to AV, QV will pass the tangency point O, as well as H (as by circle inversion in C_{HB}, C_{AB} maps on C_{DB}). One thus also finds O, as can be seen in Figure 13, as the intersection of HV and C_{DB}!

[26] For example, by constructing the centre point T_1 as the intersection point of the radius AV and the perpendicular to FG at N (and similar for T2). BV' and CV are, by the way, tangents to the circles C_{BY} and C_{CY} as BVC and BV'C are right triangles.

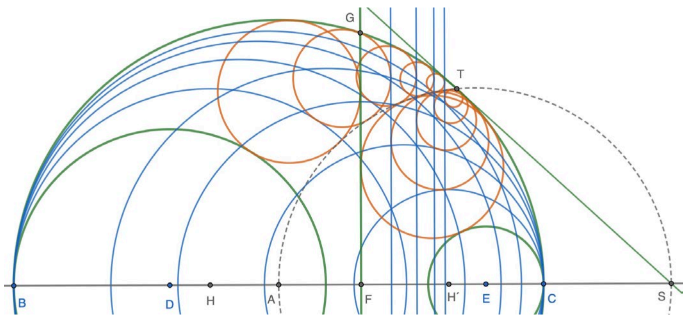
I then tried to generalise the result in Bergsten (2009) that (also) in the case of an open arbelos, the circle with diameter AS is tangent to both twin circles (Figure 14). This result means that each twin circle is tangent to three circles related to the open arbelos, as well as the power line.

Figure 14. A circle tangent to both twins in an open arbelos



Now, drawing on the work with the chain of incircles above (Figure 9), the observation in Figure 14 could indeed be generalised, as shown in Figure 15. Starting with given inner circles C_{DB} and C_{EC} and outer circle C_{AB}, successive pairs of inner magic circles of higher order were iteratively constructed (Figure 15; cf. Figure 9), along with the corresponding twin circles.[27] It was then confirmed by the drag mode that these two chains of twin circles remain tangent to the circle with diameter AS (outer tangency for the left chain and inner for the right chain). As the algebraic proof in Bergsten (2009) of this tangency property for the first pair of twins holds for any choice of r_1 and r_2 in an open arbelos, it holds for any such pair in Figure 15. So, for this generalisation, the previous work on the incircle was helpful.

Figure 15. A chain of twin circles in an open arbelos



Note. The figure shows a chain of inner magic circles, their power lines, and corresponding twin circle pairs tangent to the circle with diameter AS; FG is the power line to the given inner circles C_{DB} and C_{EC}.

[27] Here, the first order twin circles are tangent to the given inner circles (C_{DB} and C_{EC}) and their power line, while the second order twin circles are tangent to the first order inner magic circles and their power line, and so on; see Figure 15.

Following up the work on circle chains in an arbelos in Bergsten (2006, Figure 16), by further exploration with the DGE I could observe that also in an open arbelos, chains of tangent circles left and right to the power line can easily be constructed by their tangency points on the first order inner magic circles. They are all mapped on themselves by circle inversion in these magic circles. With intersecting inner circles, the chains remain similar to the situation in an arbelos. However, in the case of disjoint inner circles, the circles in the lower chains (tangent to the power line) at some point “slip out” and get bigger and bigger, eventually falling outside the outer circle. This was just fun to do.

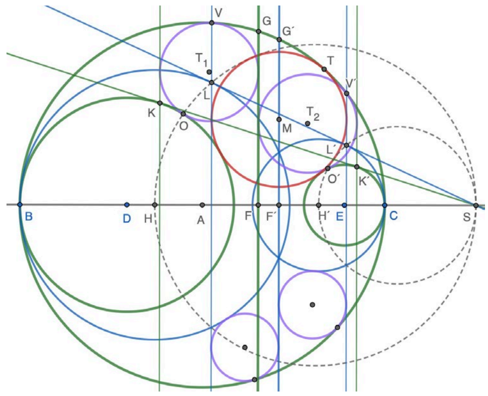
3.4 Connections between incircles and twins in an open arbelos

During the work narrated here, some theoretically interesting connections between the incircle and the twin circles in an open arbelos spoke to the eye.[28] To discover connections is indeed an educational goal in the learning of mathematics, being a key element in processes of explanation and understanding.

As discussed above, the perpendiculars to the diameter BC by the tangency points L and L’ of the common tangent to the inner magic circles are tangent to the incircle and intersect the circle C_{AB} at the tangency points V and V’ of the (first order) twin circles (Figure 6, Figure 13). This constitutes one stunning example of such a connection. In addition, it could also be noted (Figure 16) that these vertical tangents to the incircle are tangent to the second order twin circles (i.e., the circles tangent to the circle C_{AB}, the first order magic circles C_{HB} and $C_{H'C}$, and their power line F’G’), the same way as the first order twin circles are tangent to the perpendiculars to BC by K and K’! This means that the sum of the diameters of the second order twin circles equals the diameter of the (first order) incircle. This property obviously holds likewise for higher (and lower) order twin circles and incircles, as formulated in Proposition 2.

[28] In the arbelos literature, connections between twins and incircles have not been much discussed. One example is the *Bankoff triplet circle* (Bankoff, 1974), which was further elaborated in Yiu (2005).

Figure 16. Connections between incircle and twin circles in an open arbelos



Note. The second order twin circles are drawn below BC.

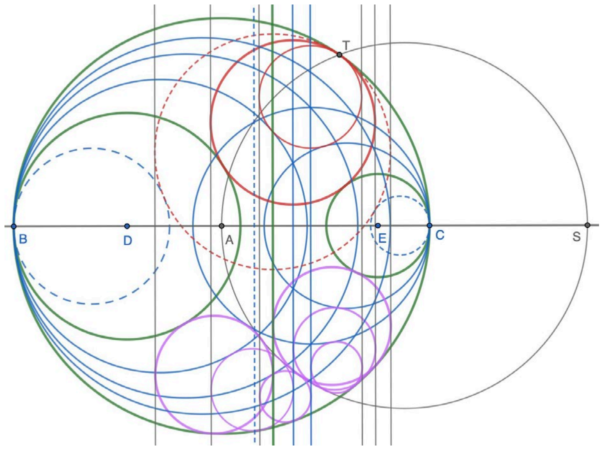
Proposition 2
For an open arbelos with given inner circles and any whole number n, the sum of the diameters of the twin circles of order $n+1$ equals the diameter of the incircle of order n, which equals the horizontal distance between the perpendiculars to the diameter BC by the tangency points of the common tangent to the inner magic circles of order n, which equals the harmonic mean of $1-b_n$ and c_n, with b_n and c_n defined as in Section 3.

Here, b_n and $1-c_n$ are the diameters of the inner magic circles of order n, where n is any whole number. By Proposition 2, the vertical tangents to an incircle, as well as the vertical line through its centre (i.e., the power line to the inner magic circles; see Figure 16), are also vertical tangents to the twin circles one order higher. Proposition 2 is further visualized in Figure 17. The incircles (shown above BC) have their centres on the power lines of the inner magic circles, and the twin circles (shown below BC) are tangent to these power lines.

The diameter of a twin circle thus equals the radius of an incircle one order lower and has the same vertical tangent. This means that the diameter d_n of a twin circle of order n (where n is any whole number) can be found by using Formula (4) in Section 3.2:

$$d_n = \frac{r_2(1-r_1)}{2(r_2(1-r_1)+2^{n-2}((1-r_1)(1-r_2)+r_1r_2))} \qquad (5)$$

Figure 17. Connections between incircles and twin circles



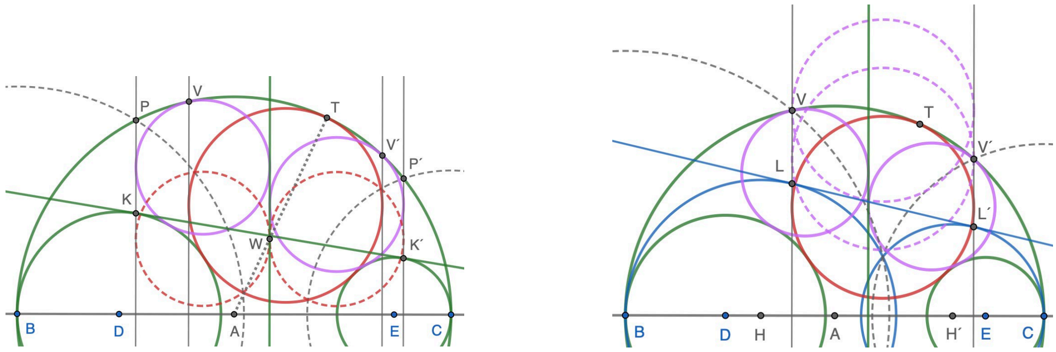
Note. The connections between incircles and twin circles by the power lines and vertical tangents are shown; lower order magic circles with power line and incircle are dotted. The given inner circles in the open arbelos with outer circle C_{AB} are C_{DB} and C_{EC} .

I could also confirm by dragging that the circles with diameters HS and H´S, both tangent to the incircle (Figure 11), intersect the circles C_{DB} and C_{EC} at the tangency points of the twin circles (i.e., points O and O' in Figure 16)! This astonishing, and for me unexpected, connection between the incircle and the twins, which can be "explained" using inversion in the circle C_{ST}, is useful for the construction of the twins. It obviously also holds for other orders of incircles/twin circles with corresponding orders of inner magic circles.

As another (indirect but nice) connection between incircles and twins, a "free" exploration with the DGE unveiled that the incircle by inversion in the magic circles C_{BP} and $C_{CP'}$, respectively, maps on circles both congruent to the twin circles and tangent to each other at the intersection W of the power line and AT (Figure 18, left)[29], while the twin circles map on themselves. "Conversely", by inversion in C_{BV} and $C_{CV'}$, respectively, the twin circles are mapped on circles both congruent to the incircle (Figure 18, right), while the incircle is mapped on itself. Noteworthy are also the positions of the mapped circles between the incircle tangents and that the incircle by inversion in C_{BV} and in $C_{CV'}$ maps on itself.

[29] As C_{AB} by inversion in C_{BP} and $C_{CP'}$ maps on the "outer tangents" to the twin circles, respectively, point T maps on the points of tangency of these new twins with these tangents. The inner circles C_{DB} and C_{EC} are by C_{BP} and $C_{CP'}$, respectively, mapped on the power line, and their tangency points with the incircle are mapped on W.

Figure 18. Indirect connections between incircle and twins

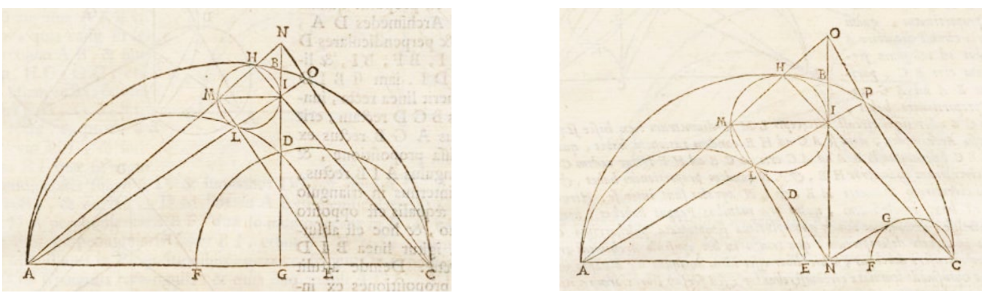


Note. The incircle inverted in C_{BP} and $C_{CP'}$ on "twins" (dotted left); the twin circles inverted in C_{BV} and $C_{CV'}$ on "incircles" (dotted right).

4 Some historical notes on the open arbelos

With my story of the construction of geometric knowledge in an open arbelos now being told, one question to ask is which parts of this knowledge were new (at the moment of its construction, and now), not only to the author but also to the mathematical community? Researching relevant literature, I noted that many recent articles dealing with this mathematical topic do not refer to older works. However, already Heath (1897, p. 307) points out, in a comment to the proof in *The Book of Lemmas* of the equality of the diameters of the arbelos twin circles, that "an Arabian Scholiast Alkauhi" had generalised Archimedes' result (i.e., to what I call an open arbelos), and that Alkauhi's proof is "similar". I found a proof in Welch (1949, p. 11-13), who referred to the work of Alkauhi[30] as it was presented in Borellus (1661) (see Figure 19).

Figure 19. From the work of Alkauhi as reported in Borellus (1661)



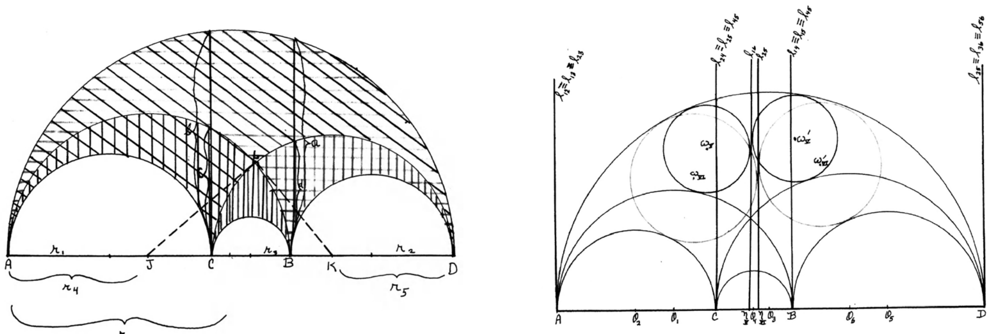
Note. Drawings of a twin circle left to the power line in an open arbelos with intersecting (left image) and disjoint (right image) inner circles, with reference to "Alkauhi" (Borellus, 1661, pp. 393-395)

[30] In some sources written Abu Sahl al-Kūhī, living in the 10th century A.D. See Berggren (2011) for an outline of the life, work, and influence of al-Kūhī.

In Figure 19 one can see several instances of applicability of the lemma in Appendix B, as used also in the proof for the congruence of the arbelos twin circles in *The Book of Lemmas*. In a more recent source, Alkauhi's proof is presented in Hogendijk (2008), who situates his work historically. What Alkauhi proved is (here with reference to notations in Figure 19, right image) that the rectangle by MI and AC is equal to the rectangle by AN and CE (cf. Hogendijk, 2008, pp. 14-15). Using the same reasoning for the right twin circle, he concluded that their diameters are equal. Now, drawing on the notations in Section 3, with $d = MI$ this would imply that $d \cdot 1 = a \cdot (1 - r_1)$, from which follows that $d = \frac{r_2(1-r_1)}{r_2+1-r_1}$. So, the harmonic mean formula for the diameter of the open arbelos twin circles (as in Bergsten, 2009, and Section 5 above)[31] is indeed implicit in Alkauhi's proof. This type of observation is educationally notable as an example of the "connectedness of mathematics" (Kondratieva & Bergsten, 2021) across time and mathematical discourses.

Welch (1949, pp. 10-13) further discussed what she termed "harmonic arbeloi", referring to the four arbelos drawings that can be identified in the left image in Figure 20, based on a harmonic range of points A, C, B, and D on a line. Letting the (semi)circles on AD, AC and BD correspond to the (semi)circles C_{AB}, C_{DB} and C_{EC} in Figure 4, the (semi)circles on AB and CD (happen to) coincide with what I call the first order (inner) magic circles to the circles AC and BD[32]. Welch also reads the right image in Figure 21 in terms of what I call an open arbelos with inscribed twin circles, also providing formulas for their radii (p. 88).

Figure 20. "Harmonic arbeloi" in Welch (1949, Figure 37 and Figure 39, n.p.)



Further, in his list of properties of the "marvelous arbelos", Bankoff (1994, pp. 252-253) presents images of twin circles in an open arbelos as "a variation of the property regarding the equality of the twin circle", without reference or further elaborations.

Taking a mathematically more general approach than in this paper, going beyond school level mathematics, Okumura and Watanabe (2008, 2009) investigated incircles

[31] An equivalent formula can also be found in Okumura (2012, p. 18).
[32] This happens exactly when the point sequence ACBD is a harmonic range. In my notation for the open arbelos, this occurs when r_2 is the harmonic mean of 1 and r_1, which then equals the diameter of the inner magic circle C_{HB} to C_{AB} and C_{DB} (see Section 3.1); similar for the inner magic circle $C_{H'E}$ to C_{AB} and C_{EC}. In Welch (1949) I could not find an explicit argument for looking specifically at "harmonic arbeloi".

and twins (and their radii) in their configuration of a "generalized arbelos in aliquot parts", which also includes what I call an open arbelos. Constructions were not discussed.

While investigating a generalization of Archimedes' *Salinon*, Ciaurri and Fernández (2017) determined the radius of the incircle in an open arbelos with disjoint inner circles. Their relational formula easily translates into the direct formula for ρ in my notation in Section 5. They also proved the equality of the twin circles in an open arbelos with disjoint inner circles and determined its diameters as (translated to my notation) half the harmonic mean of r_2 and $1-r_1$ (pp. 361-363); here they mention the old results of "Alkauhi". The authors did not address the case of intersecting inner circles or constructions of the incircle / twin circles.

With these short commented historical remarks, I wanted to set my narrative in a wider mathematical and historical perspective. Considering the vast (also educational) literature on the arbelos, the open arbelos has attracted much less attention (especially in education).

5 Discussion and conclusion

Even if the aim of taking a narrative approach in Section 3 was not to give the readers "the illusion of a mathematics created under their very eyes" (Freudenthal, 1973, p. 116), the text may to some extent provide a genetic trace of the constructed geometric knowledge, and of the role of the DGE in this process. The story was construed as a sequence of clauses, presenting its plot within a mathematical context of circle tangencies in an open arbelos, aiming at constructions and proofs of properties of incircles and twin circles (cf. Mor & Noss, 2008). As to the four "characteristics" attached to a mathematical narrative suggested by Healy and Sinclair (2007), here embedded within a purely mathematical discourse, *sequentiality* of clauses moved the story forwards. The *imaginary* appeared, for example, as magic circles constructed "inwards" as a contrast to the *real* magic circles that were defined to be constructed "outwards". Unexpected or *exceptional* connections between incircles and twin circles could be drawn from the *ordinary* tangency properties that were constructed. A kind of *dramatic quality* can possibly be seen in the quest of finding new constructions and properties, and in some of the unexpected "discoveries". The narrative thus expressed how the *ideational mathematics* (cf., Section 2.3) of the author developed along with the DGE investigations, and was formulated in the institutionalised mathematical discourse. By the educational gaze taken, a discovery learning sequence could be followed to uncover a detailed knowledge construction process.

From this narrative and the historical remarks made, it can be concluded, as an answer to the question about new mathematical knowledge raised at the beginning of Section 4, that (to the author's current knowledge) the following geometric knowledge constructed in Section 3 is new:

1. In an open arbelos, the incircle centre is on the power line to the inner magic circles (Figure 6) and the centres of the twin circles are on the power lines to the outer

magic circles (Figure 12).[33] These properties can be generalised to all orders of incircles/twin circles/magic circles.

2. The connections between diameters of incircles and twin circles, as presented in Proposition 2, along with the formulas for the incircle and twin circle radius of different orders, as well as for the inner magic circles of different orders.[34] Specifically, the diameters of the twin circles and incircles can be expressed as harmonic means of $1 - r_1$ and r_2, and $1 - r_1'$ and r_2', respectively (similar for other orders).
3. The tangency points of the twin circles to the open arbelos outer circle are where the vertical tangents to the incircle intersect the outer circle (similar for all orders of incircles/twins). Vertical tangents to twin circles[35] coincide with vertical tangents to incircles one order lower.
4. The circles with diameters HS and H´S are tangent to the incircle (Figure 11) and pass the tangency points of the twin circles to the given inner circles (Figure 16). This property can be generalised to all orders of incircles and twins.[36]
5. The circle with diameter AS is tangent to the twin circles (Figure 14). This property holds for all orders of twin circles (Figure 17).
6. The segments HT and H´T intersect the given inner circles at the tangency points of the incircle to these circles (Figure 6). This property can be generalised to all orders of incircles.[31]
7. Most of the constructions discussed (see Appendix D for an example).

So, explorative work using a DGE on a geometric topic or theme did lead to "discoveries" of invariances by drag mode tests drawing on figural concepts, thereby establishing a basis for "new" knowledge. Explanatory efforts here included reasoning with figural concepts, geometric proof (Euclidean), algebraic verification, or at another level "empirical confirmation" afforded by the algebraic representations of conceptual relationships. Alternative geometric constructions of key objects based on the establishment of new knowledge could be developed. I would suggest that all these educational elements are potentially there also at students' explorative work using a DGE in school geometry at different levels (cf., Baccaglini-Frank, 2019. Laborde et al., 2006; Sinclair & Robutti, 2013).

In addition, algebraic explorations led to the establishment of geometric constructions of objects (indeed common in geometry), the conceptual properties of which could be confirmed and further developed by observations of invariances in the DGE drag mode.

[33] Thus located, more specifically, at the intersections of these power lines with the radius of the open arbelos outer circle to the tangency points of the incircle/twin circles. Note that C_{BP} is a midcircle to C_{AB} and the line by K perpendicular to BC, that $C_{CP'}$ is midcircle to C_{AB} and the line by K' perpendicular to BC, and that C_{CG} and C_{BG} are midcircles to C_{AB} and the power line by FG (see Figure 12)

[34] For the incircle and twins, see formulas (4) and (5). The diameters of the inner magic circles follow from (3).

[35] That is, left tangent for the left twin and right tangent for the right twin. The common tangent to the twins is the power line to the magic circles one order lower. Generally, the vertical tangents (other than the power line) have, to my knowledge, not been much discussed in the arbelos literature.

[36] H and H' (i.e., the centres of the first order inner magic circles) must then be read as the centres of inner magic circles of corresponding orders.

One example is Proposition 1, where algebraic analyses were used as underpinnings for key constructions – one could say as agents making things happen – as well as verification tool. This latter would certainly apply to school geometry for the purpose of demystifying constructions that for a student may seem "impossible" to discover.[37]

As another key observation (or maybe even a "result"), it was pointed out in Section 3 that the formulas in the open arbelos case for the radii of the incircle and the twin circles are "direct" generalisations of the corresponding formulas in the arbelos case. Within a DGE, this correspondence lends itself to dynamic explorations, drawing on (for example) the constructions in Appendix D. By dragging the points B' and/or C' (notation as in Figure 2), and highlighting the corresponding formulas, a user can follow the continuous change of the radii of the incircle and/or the twin circles from the open arbelos case (disjoint inner circles) to the arbelos case, and then again to the open arbelos case (intersecting inner circles).

Exploration in a DGE is by necessity guided by the prior knowledge of the explorer(s), setting limits to its scope. However, similar "discoveries" can certainly be made drawing on different knowledge bases. One example is provided by what I call magic circles. In the preparation of Bergsten (2006) they were, as indicated in Section 2 above, simply tried out and found to be not only useful but also fundamental for the construction of tangent circles in an arbelos. Independently, Danneels and Lamoen (2007) identified the same circles in the arbelos by conceptualizing them theoretically as circles which by circle inversion map two given circles on each other (i.e., midcircles). Generalising the arbelos to an open arbelos, the narrative presented here has shown that the "corresponding" magic circles remain fundamental for the circle tangencies, also revealing fundamental connections between incircles and twin circles. The concept of midcircles could be referred to for "explaining" some of the observations and constructions, thus serving an educational purpose. Explorations within a DGE certainly benefit from being connected to a larger knowledge base (cf. Hofstadter, 1997). Therefore, similar exploration activities can constructively be used at different school levels, as well as in individualised teaching.

The discussion of the literature search presented in Section 4 also showed that some of the geometric knowledge related to an open arbelos presented in this paper, which was new to the author at the moment of its construction, was not new to the mathematical community (e.g., the equality of the twin circles in an open arbelos). This mirrors the situation for a student in a mathematics classroom engaged in discovery learning. Indeed, the personal experience of a discovery is potentially there in any problem-solving activity, be it in a specific or a general context. In geometry, providing a DGE for individual or collaborative work on explorative but guided well-designed activities may support the generation of students' own stories (mathematical narratives) as pathways to their construction of geometric knowledge.[38]

[37] As examples from my own experience, I could mention constructions of a pentagon (e.g., Bergsten, 2003) and points on a parabola (e.g., Bergsten, 2015; Kondratieva & Bergsten, 2021).
[38] One example of an explorative guided DGE student activity is shortly sketched in Appendix E.

Research ethics

Artificial intelligence

Artificial intelligence has not been used for or in writing this article.

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Conflicts of Interest

The author declares no conflicts of interes

References

Amaral-Schio, R. B., & Haug, R. (2020). Possibilities of the DGE use in Math class: Brazilian and German experiences. *Revista Sergipana de Matemática e Educação Matemática (REVISEM)*, *5*(1), 1–28. https://doi.org/10.34179/revisem.v5i1.12192

Archimedes (n.d./2018). *Book of lemmas*. Translated by N. L. Kechris. https://archive.org/details/enbibliolimmaton/page/n1/mode/2up

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, *34*(3), 66–72. https://doi.org/10.1007/BF02655708

Baccaglini-Frank, A. (2019). Dragging, instrumented abduction and evidence, in processes of conjecture generation in a dynamic geometry environment. *ZDM Mathematics Education*, *51*, 779–791. https://doi.org/10.1007/s11858-019-01046-8

Bankoff, L. (1994). The marvelous arbelos. In R. K. Guy & R. E. Woodrow (Eds.), *The lighter side of mathematics: Proceedings of the Eugène Strens Memorial Conference on Recreational Mathematics and its History* (pp. 247–253). Mathematical Association of America, Washington, DC.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843-908). National Council of Teachers of Mathematics.

Berggren, J. L. (2011). Lost or embedded works of Ku¯hi¯. *Tarikh-e Elm*, *9*, 1–19.

Bergsten, C. (2003). En gyllene pyramid. Fem trianglar och en pentagon [A golden pyramide: Five triangles and a pentagon]. *Nämnaren*, 30(2), 36–41.

Bergsten, C. (2006). Magic circles in the arbelos. Report LiTH-MAT-R-2006-12, Department of Mathematics, Linköping University. Later published in *The Mathematics Enthusiast*, *7*(2&3), 209–222. https://scholarworks.umt.edu/cgi/viewcontent.cgi?article=1184&context=tme

Bergsten, C. (2009). Tvillingcirklar [Twin circles]. *Normat* (*Nordisk Matematisk Tidskrift*), *57*(1), 22–31. https://normat.ncm.gu.se/2009.html

Bergsten, C. (2015). Beyond the representation given. The parabola and historical metamorphoses. In C. Bergsten & B. Sriraman (Eds.), *Refractions of mathematics education* (pp. 15–47). Information Age Publishing.

Borellus, A. (Ed.) (1661). *Apollonii Pergaei Conicorum, Lib. V. VI. VII, Archimedis Assumptorum Liber*. Florentiae MDCLXI. https://www.e-rara.ch/zut/content/zoom/2585234

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review*, *31*, 21–32.

Ciaurri, Ó., & Fernández, E. (2017). Variations on an Archimedean ground: The generalized Salinon. *The College Mathematics Journal*, *48*(5), 355–365.

Coxeter, H. S. M., & Greitzer, S. L. (1967). *Geometry revisited*. The Mathematical Association of America.

Danneels, E., & van Lamoen, F. (2007). Midcircles and the arbelos. *Forum Geometricorum*, *7*, 53–65.

Danzer, C. (2024). Attitudes in mathematical discovery processes: The case of Alex and Milo. *LUMAT: International Journal on Math, Science and Technology Education*, *12*(1), 98–112. https://doi.org/10.31129/LUMAT.12.1.2131

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139–62. https://doi.org/10.1007/BF01273689

Freudenthal, H. (1973). *Mathematics as an educational task*. D. Reidel Publishing Company.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometric constructions and proving activities: A case of cognitive unity? In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, 9–16). PME.

Goldenberg, E. P., & Cuoco, A. A. (1998). What is dynamic geometry? In R. Lehrer, & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 351–367). Lawrence Erlbaum Associates.

Healy, L., & Sinclair, N. (2007). If this is our mathematics, what are our stories? *International Journal of Computers for Mathematical Learning*, *12*, 3–21.

Heath, T. L. (Ed.) (1897). *The works of Archimedes. Edited in modern notation with introductory chapters*. The University Press. https://www.aproged.pt/biblioteca/worksofarchimede.pdf

Hofstadter, D. R. (1997). Discovery and dissection of a geometric gem. In J. R. King & D. Schattschneider (Eds.), *Geometry turned on! Dynamic software in learning, teaching, and research* (pp. 3–14). The Mathematical Association of America.

Hogendijk, J. P. (2008). Two beautiful geometric theorems by Abu¯Sahl Ku¯hi¯in a 17th century Dutch translation. *Ta¯ri¯kh-e 'Elm: Iranian Journal for the History of Science*, *6*(1), 1-36.

Kondratieva, M. (2013). Geometrical constructions in dynamic and interactive learning environment. *Mevlana International Journal of Education*, *3*(3), 50–63. (Special Issue: Dynamic and Interactive Learning Environment) https://files.eric.ed.gov/fulltext/ED544152.pdf

Kondratieva, M., & Bergsten, C. (2021). Secondary school mathematics students exploring the connectedness of mathematics: The case of the parabola and its tangent in a dynamic geometry environment. *The Mathematics Enthusiast*, 18(1–2), 183–209. DOI: https://doi.org/10.54870/1551-3440.1520

Laborde, C. (1993). The computer as part of the learning environment: The case of Geometry. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp. 48–67). Springer.

Laborde, C., Kynigos, C., Hollebrands, K., & Sträßer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275–304). Sense Publishers.

Lachmy, R. & Koichu, B. (2014). The interplay of empirical and deductive reasoning in proving "if" and "only if" statements in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *36*, 150–165.

Leuders, T., & Philipp, K. (2013). Preparing students for discovery learning – skills for exploring mathematical patterns. In A. M. Lindmeier & A. Heinze (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3*, (pp. 241-248). Kiel, Germany: PME.

Leung, A. (2015). Discernment and reasoning in Dynamic Geometry Environments. In S. J. Cho (Ed.), S*elected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 451–469). Springer International Publishing. DOI 10.1007/978-3-319-17187-6_26

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics*, *44*, 25–53. https://doi.org/10.1023/A:1012733122556

Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics*, *34*, 219–248. https://doi.org/10.1023/A:1002985109323

Massarwe, K. H. (2023). Studying geometric concepts in elementary school through construction by compass and straightedge. *International Journal on Studies in Education (IJonSE)*, *5*(1), 42–63. https://doi.org/10.46328/ijonse.95

Mor, Y., & Noss, R. (2008). Programming as mathematical narrative. I*nternational Journal of Contiuing Engineering Education and Life-Long Learning*, *18*(2), 214-233. hal-00591784f

Okumura, H. (2012). Ubiquitous Archimedean circles of the collinear arbelos. *KoG*, *16*, 17–20.

Okumura, H., & Watanabe, M. (2008). Generalized arbelos in aliquot parts: Intersecting case. *Journal for Geometry and Graphics*, *12*(1), 53–62.

Okumura, H., & Watanabe, M. (2009). Generalized arbelos in aliquot parts: Non-intersecting case. *Journal for Geometry and Graphics*, *13*(1), 41–57.

Oller-Marcén, A. M. (2016). Archimedes' Arbelos to the n-th dimension. *Forum Geometricorum*, *16*, 51–56.

Różański, M., Samulewicz, A., Szweda, M., & Wituła, R. (2017). Variations of the arbelos. *Journal of Applied Mathematics and Computational Mechanics*, *16*(2), 123–133.

Sinclair, N., & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In K. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 571–596). Springer.

Takaya, K. (2008). Jerome Bruner's Theory of Education: From early Bruner to later Bruner. *Interchange*, *39*(1), 1–19.

Welch, G. (1949). *The arbelos*. Master Thesis. Department of Mathematics, University of Kansas. Available from CORE at https://core.ac.uk/reader/213409760

Yiu, P. (2005). Elegant geometric constructions. *Forum Geometricorum*, *5*, 75–96.

Appendix A

A straightforward algebraic calculation provides the following expressions for the coordinates (u, v) of the point of tangency T for the tangent from S to C_{AB} (Figure 5):

$$\begin{cases} u = \dfrac{r_1 r_2}{(1-r_1)(1-r_2)+r_1 r_2} \\ v = \dfrac{\sqrt{r_1 r_2 (1-r_1)(1-r_2)}}{(1-r_1)(1-r_2)+r_1 r_2} \end{cases}$$

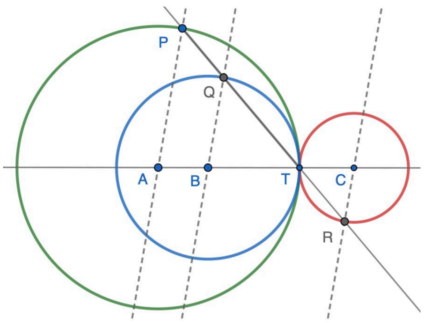
A direct calculation then confirms that the point (u, v) is on the line through A and M (Figure 5), thus showing that the incircle is tangent to the circle C_{AB} at point T.

Appendix B

The lemma on circle tangency illustrated in Figure 21 was presented as Proposition I in *The Book of Lemmas*, where the case of inner tangency of the circles was proved. In Figure 21 both cases are highlighted, as they are used for constructions in this paper.

The circles C_{AT} and C_{BT} are tangent internally at T, and the circles C_{AT} and C_{CT} are tangent externally at T. This means that T is on the line by AB and by AC. P is an arbitrary point on C_{AT} (not on the diameter through A and T). The line by B parallel to AP meets the circle C_{BT} at Q. The Lemma then states that the point T is on the line by P and Q, and when the line by C parallel to AP meets the circle C_{CT} at R, the point T is on the line by P and R.

Figure 21. An illustration of Proposition 1 in *The Book of Lemmas*



Appendix C

In Figure 11 the point M_1 is the centre of the circle with diameter HS and M_2 the centre of the circle with diameter H'S. Assuming the incircle C_{MT} and the circle C_{M_1S} are tangent, the application of Pythagoras' Theorem to the triangle M_1MF' would give the following equation:

$$\left(\frac{r_1(1-r_1+r_1r_2)}{2(1+r_1)(r_1+r_2-1)} - \frac{r_2(1-r_1)}{2(1-r_1+r_1r_2)}\right)^2 =$$

$$= \frac{r_1r_2(1-r_1)(1-r_2)}{(1-r_1+r_1r_2)^2} + \left(\frac{r_1(r_1+2r_2+r_1r_2-1)}{2(1+r_1)(r_1+r_2-1)} - \frac{r_2(1+r_1)}{2(1-r_1+r_1r_2)}\right)^2$$

With reference to Figure 11, assuming the incircle C_{MT} and the circle C_{M_2S} are tangent, the application of Pythagoras' Theorem to the triangle M_2MF' would give the following equation:

$$\left(\frac{2r_1r_2 - r_1r_2^2 - r_1 - r_2 + 1}{2(2-r_2)(r_1+r_2-1)} + \frac{r_2(1-r_1)}{2(1-r_1+r_1r_2)}\right)^2 =$$

$$= \frac{r_1r_2(1-r_1)(1-r_2)}{(1-r_1+r_1r_2)^2} + \left(\frac{r_1r_2}{2(r_1+r_2-1)} + \frac{1}{2(2-r_2)} - \frac{r_2(1+r_1)}{2(1-r_1+r_1r_2)}\right)^2$$

That these equations actually hold is most easily confirmed using a CAS.

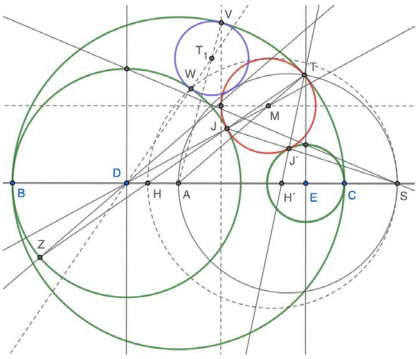
Appendix D

A construction of the incircle in an open arbelos is described below, followed by a construction of a twin circle (see Figure 22). To save space only the constructed geometrical objects will be named or shortly described in the order of the construction steps, with notations as throughout this paper. Given is an open arbelos with outer circle C_{AB} and inner circles C_{DB} and C_{EC} .

Incircle: S, T, AT, line by D parallel to AT intersects C_{DB} at Z, TZ intersects BC at H and C_{DB} at tangency point J of incircle, line by D and J intersects AT at center point M of incircle

Left twin circle: circle with HS as diameter intersects C_{DB} at tangency point W of twin circle, left vertical tangent to incircle C_{MT} intersects C_{AB} at tangency point V of twin circle, line by DW intersects AV at center point T_1 of twin circle

Figure 22. Constructions of incircle and left twin circle



Note. Full construction lines are used for the incircle, while dotted construction lines are indicated for the left twin circle (to construct the right twin circle, the circle with diameter H'S can be used in a similar way).

Appendix E

The construction of a tangent to a circle from a given point outside the circle is commonly included in geometry teaching at upper secondary school. This construction can form a basis for further explorations (cf., Kondratieva, 2013) leading to, for example, circle inversion and the notion of a power line. In the following DGE activity (for students working in pairs, having prior experience with DGE constructions), the goal is to learn how to construct a common tangent to two circles (which could lead to the notion of a power line, as indicated in Section 3.1 above). What narratives may students develop in this activity?

The "success" of this DGE activity draws on observations that students indeed notice invariances when dragging, in line with the *transformational-saliency hypothesis* (Battista, 2007) which implies that "it is not just that one might see invariance in dragging but that one cannot help but notice it" (Sinclair & Robutti, 2013, p. 577).

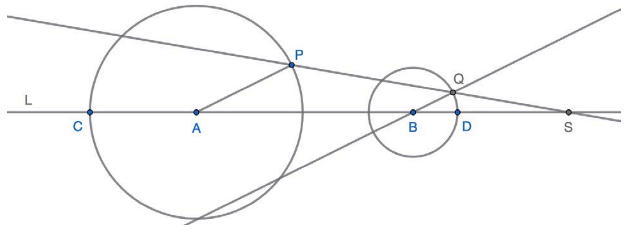
Sketch of an explorative geometry DGE activity for students

You already know how to construct a tangent to a circle from a point outside the circle. Carry out that construction with a DGE, confirm that it works by using the drag mode, and explain why the construction works. Then continue to the following problem.

Problem: Construct a straight line that is tangent to two given circles.

DGE activity: Discuss and explore how a solution to this problem could look like before trying out the following activity:

On a straight line L, select two distinct points A and B, and draw a circle (C_{AC}) with centre point A intersecting L at C, and a second circle (C_{BD}) with centre point B intersecting L at D (as in the figure below). Select a point P on C_{AC} and draw the radius AP as well as a line by B parallel to AP, intersecting C_{BD} at Q. Finally, draw a line passing P and Q intersecting L at S.



Now, use the drag mode to move the point P along the circle C_{AC} and look what happens. Is there some observation that speaks to the eye? Can you make the same observation with different sizes and positions of the two circles?

1. How can you explain the observation of the point S?
2. How can you use the point S to construct a line that is tangent to both circles?
3. Suggest a suitable name for the point S !

The line passing B and Q intersects the circle C_{BD} also at the point Q' (as in the figure below). Draw a line passing P and Q' intersecting L at S'. Move the point P on the circle C_{AC} and observe what happens. Then consider again the questions 1. and 2., now for the point S'.