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Mapping meaning to mathematics in teaching single-photon quantum optics experiments to pre-service physics teachers

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Abstract: In this study, I discuss a didactical approach to how single photon quantum optical experiments can be described by using Dirac's bra-ket notation to translate the meaning of experiments to pertinent mathematical structures of quantum states. The exploratory research reported here was conducted as a part of pre-service physics teachers' introductory course on quantum physics emphasizing the quantum optical approach. The data was gathered from the weekly course assignments and from the course exam. The design of the assignments was inspired by the epistemic game of mapping meaning to mathematics (MMM). The students' responses were analysed using a rubric analysis. The results show that, within this course and tasks, participants generally produced correct quantum state-evolution using Dirac's bra-ket notation as a mapping, and probability calculations with the Born rule in the context of quantum optical interferometric experiments. The results suggest that implementing bra-ket formalism through MMM may provide students with a symbolic framework for describing quantum state evolution and measurement outcomes.

Keywords: bra-ket notation, Born rule, quantum education, quantum optics experiments, pre-service physics teachers, epistemic games

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1. Introduction

Quantum physics and the second revolution of quantum technologies are becoming increasingly relevant and visible in society. Consequently, physics teachers need to be prepared with a solid understanding of the fundamental physical principles and concepts that underpin these developments. Physics teachers play a key role in inspiring new quantum physicists and can convey quantum physical perspectives to a wide audience of students. Mathematical formalism plays a central role in understanding such a highly mathematical theory as the quantum theory of light. Many of its central concepts are essentially mathematical concepts derived from the underlying mathematical structures. In learning quantum physics, the correct use of mathematical formalism is often perceived as a challenge for students (Bouchée et al., 2022; Bungum et al., 2015; Pereira & Solbes, 2022). In

particular, making connections between physical reality and mathematical formalism (Bouchée et al., 2022) as well as connecting the conceptual reasoning (e.g., in making meaning of experiments) to mathematical formalism, appear to be challenging for students (Bouchée et al., 2022; Myhrehagen & Bungum, 2016). Nevertheless, understanding the mathematical foundations of quantum theory is needed to some extent even in teaching quantum physics at introductory and secondary level (Merzel, Bitzenbauer, et al., 2024). However, research-based evidence shows that there are ways of implementing the necessary mathematical structures in upper-secondary level teaching. Therefore, it is important to find ways in which the key role of mathematical structures in quantum physics can be addressed in pre-service physics teacher education.

Recent studies have suggested that using Dirac's bra-ket notation instead of the wave function or matrix representations facilitates learning the mathematical structure of quantum physics (Gire & Price, 2015), for example the identification of probability amplitudes, quantum coherence, and quantum states (Gire & Price, 2015; Hennig, Tóth, Förster, et al., 2024). At the university level, Dirac notation has often been approached with considerable mathematical rigour. Studies report graduate and undergraduate students' difficulties with states, operators, and probabilities of particular outcomes (Singh & Marshman, 2013), including observables (Marshman & Singh, 2018). Lautesse et al. (2026) report that undergraduate students are more proficient at recognising state vectors and eigenvectors than constructing them themselves. They also report that students exhibit confusion between position and momentum bras and the corresponding operators, and confusion with Dirac delta and Kronecker delta functions. In order to reduce mathematical complexity and generate intuition regarding the mathematics, Levy & Singh (2025) proposed an undergraduate introductory approach that describes two-state systems first with familiar unit vectors after extending to what they call "dot-vectors" before introducing Dirac's bra-ket notation and finally extending to the Born rule. This was contextualised in what they call a "Bloch cube", a more accessible version of the Bloch sphere.

In the context of quantum optics experiments, mathematical elements provide the basis for mapping experimental systems to the mathematical structures of quantum states, and further, in calculating detection probabilities by using the Born rule.

Marshman & Singh (2016) exposed upper-level undergraduates and PhD students to Mach-Zehnder interferometer with polarisers through quantum interactive learning tutorials. Their study revealed students' problems in identifying second beamsplitters effects on measurement outcomes, polarisers, and polarisation states as measurement bases. Some recent studies have addressed students' use of Dirac notation in quantum optical contexts at upper-secondary level (Hennig, Tóth, Förster, et al., 2024; Hennig, Tóth, Veith, et al., 2024b, 2024a; Merzel, Weissman, et al., 2024). These studies show that reduced bra-ket notation might help upper-secondary students to overcome barriers in conceptual reasoning (Hennig, Tóth, Veith, et al., 2024b) as well as mathematical sensemaking in problem-solving (Merzel, Weissman, et al., 2024). Pospiech et al. (2021) implemented reduced Dirac notation with upper-secondary students in the context of

Stern-Gerlach simulation, with most of the students succeeding in using it and allowing them to make conceptual connections.

In what follows, I propose a didactical approach for introductory undergraduate teaching of pre-service physics teachers that uses Dirac's bra-ket formalism to map the quantum state of single-photon interferometric experiments. The approach is inspired by *Mapping Meaning to Mathematics* (MMM) (Tuminaro & Redish, 2007) and discussed in more detail in section 2. In this approach, mathematical representation describing the quantum state evolution is connected explicitly to quantum optical experimental situations. Following the ideas of Scarani (Scarani et al., 2010; Scarani & Suarez, 1998) and Gerry & Knight (2004), bra-ket notation is introduced as a kind of a "bookkeeping" or "mapping", tracking how optical components affect the quantum state evolution and further connecting the quantum state to measurement probabilities via the Born rule. The focus is on bra-ket notation as symbolic description of quantum states: how it is used to track the transformations of states in relation to the experimental device, from input states to output states, and eventually, how output states are related to observable outcomes through the Born rule. In this approach, students do not need advanced mathematical skills: instead, basic-level linear algebra is enough (e.g., to understand the notion of orthogonality and the use of the conjugate rule or Euler's formula) alleviating much of the mathematical complexity. Scientific rigour is maintained, while mathematical complexities are minimised. Even without complex calculations, bra-ket notation provides access to a formal framework in which concepts – such as quantum state, coherence, superposition, and entanglement – are related to the experimental configuration. This enables students to connect the symbolic and formal structure of the bra-ket notation to the operational meaning of superposition and entanglement as they arise from the arrangement and relational structure of components in interferometric quantum optics experiments. The focus of this approach is on the introductory undergraduate level so, compared to the reduced Dirac notation by Hennig, Tóth, Förster, et al. (2024), the implementation presented here preserves greater complexity of the mathematical elements while approaching Dirac notation with substantially less mathematical manipulation and calculation complexity than in the previous research settings by Lautesse et al. (2026) , Marshman & Singh (2018), and Singh & Marshman (2013). For physics teacher education, the simple two-state systems of quantum optics experiments provide demystifying examples of quantum physics by combining abstract formalism with experimentally testable phenomena (Scarani et al., 2010).

Relatively little is known about how pre-service physics teachers implement bra-ket notation to represent quantum state evolution and detection probabilities in quantum optics experimental contexts. Scant research has been done on the use of Dirac's bra-ket notation as cross-representational mapping from schematic diagram to mathematics in the context of quantum optics experiments, where students need to rely on a schematic diagram of the experiment to map the quantum state evolution and calculate detection probabilities. Research on different representation schemes often investigates the effectiveness between Dirac notation and wave function (Riihiluoma et al., 2025) and

matrix representation (Gire & Price, 2015) but does not discuss what representation schemes students prefer when describing quantum optics with bra-ket notation. To address these research gaps, the study presented here evaluates the suggested teaching approach as a part of an introductory course on quantum physics that focused on contemporary quantum optics for pre-service physics teachers (see more detailed description of the course contents Koponen et al., 2025). Quantum optics provides simple two-state systems Empirical data for evaluating the learning outcomes was gathered from weekly assignments and the course exam. In these assignments, participants were provided with a schematic diagram of a quantum optical experiment and asked to map the quantum state of the system using Dirac's bra-ket notation and to calculate detection probabilities using the Born rule. The study also contributes to recent efforts to make Dirac bra-ket notation more approachable in introductory-level teaching, with the intention being to bridge mathematical structures, concepts, and experiments without difficult mathematical calculations. This could serve as a base to learn more complex mathematics of quantum physics.

The research questions posed here are:

R1. To what extent do students correctly map quantum state evolution into bra-ket notation and calculate detection probabilities using the Born rule when interpreting schematic diagrams of interferometric quantum optics experiments?

R2. What recurrent error patterns can be identified in students' quantum state mappings and probability calculations?

R3. Which quantum-state representation schemes did students use in bra-ket notation, and how were these associated with performance level?

Moreover, I discuss the generalisation of the MMM epistemic game (Tuminaro & Redish, 2007) in the context of these quantum optics assignments by considering the noted importance of the role played by schematic description of the experimental situation. The results of this study show that pre-service physics teachers performed well when engaging with these MMM assignments throughout the course.

2. Design of the assignments: the viewpoint of Mapping Meaning to Mathematics

In this section, I will first give an overview of basic ideas on how the epistemic game (e-game) of mapping meaning to mathematics (MMM) is motivated. Second, I will outline "the meanings" in the context of single photon interferometric experiments to be mapped, and what kinds of mathematical structures they are to be mapped onto. Third, I introduce a version of the MMM e-game as a viewpoint for the design of assignments that are

contextualised in mapping the state evolution of a quantum state in quantum optics experiments as a schematic-symbolic mapping, leading to the calculation of detection probabilities using the Born rule. Here the students' goal is to use a schematic diagram of a quantum optical interferometric experiment to map the evolution of the quantum state in that system to a symbolic form using Dirac's bra-ket notation and then apply the Born rule to calculate the detection probabilities.

2.1 Epistemic game of "Mapping meaning to mathematics"

Tuminaro & Redish (2007) introduced e-games in the physics education context, particularly in the problem-solving context of "understanding and use of mathematics in the context of physics". They describe e-games as "a coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem". Previously, Collins & Ferguson (1993) defined e-games as "structures and strategies that guide inquiry", where a set of moves satisfies the inquiry. In this way, the epistemic nature of the e-games is captured in their role of encompassing the construction of new knowledge with the goal of making sense of phenomena in the world (Collins & Ferguson, 1993). Tuminaro & Redish (2007) point out the physics problem-solving activities' resemblance to a "game" (hence the name e-games) where the similar identifiable ontological features (e.g. player, game board vs. concepts, equation) and structural features (e.g. beginning, end, moves, rules) together distinguish them from other activities. Tuminaro & Redish (2007) describe moves as a defining feature of an e-game, as each e-game has a certain set of moves, while also underlining the importance of moves that are excluded. Moves represent all the actions occurring between entry and exit conditions. Entry and exit conditions comprise the conditions under which a student recognises the beginning and end of playing an e-game. Ontological components include the knowledge base and epistemic form. The knowledge base consists of the collection of all the cognitive resources that students utilise when playing an e-game, for example content knowledge, concepts, mathematical skills, and epistemology. Epistemic form is the "target structure that guides inquiry", for example flow charts and equations. (Tuminaro & Redish, 2007). The theoretical framework of epistemic games has been applied in many science education research contexts (Bajracharya & Thompson, 2016; Bielaczyc & Kapur, 2010; Chen et al., 2013; Hu et al., 2019; Tuminaro & Redish, 2007). In this paper, I will focus on the e-game of mapping meaning to mathematics, covered in greater detail below and later in the context of quantum optical experiments. Of the six different e-games identified by Tuminaro & Redish (2007), "Mapping Meaning to Mathematics" was characterised as the most complex. They divide MMM into five basic moves, where a student:

1) develops a story about a physical situation,
2) translates quantities in this story into mathematical entities,
3) relates mathematical entities in accordance with the physical story,
4) manipulates the symbols,

5) evaluates and interprets the story.

Tuminaro & Redish (2007) state that in move 1, the conceptual understanding of a student is central. In moves 2 and 3, the focus is on intuitive mathematics knowledge, symbolic forms, and interpretive devices (cognitive tools for obtaining information from mathematical formalism (Tuminaro & Redish, 2005)). In move 4, the symbolic manipulation of previously formulated equations takes place, and in move 5 students compare the quantitative answer generated to match their conceptual story or compare it to some outer source. This fifth move also serves as grounding for the exit condition where students decide whether the task is complete or not. In MMM, the most dominant epistemic form was the mathematical expressions generated in moves 2 and 3, which serve as guidance for the inquiry supported by the interpretation of the story developed (Tuminaro & Redish, 2007).

2.2 Mathematical structures for quantum interferometric experiments

The mathematical structures that are involved in the mappings of interest here are the structures that are relevant to modern theory of quantum optics and quantum theory of light (Gerry & Knight, 2004; Scarani et al., 2010; Scarani & Suarez, 1998). The contemporary description of a photon as a quantum state is formulated through a mathematical structure using Dirac's bra-ket notation. In this framework, the quantised degrees of freedom of the electromagnetic field are represented by a quantum state of the field modes in the Fock state (number state) basis, where $|1\rangle$ denotes a single photon state and $|0\rangle$ denotes the vacuum state. In the context of quantum optics experiments, e.g. Grangier et al. (1986) experiments or Which-way experiments (Maries et al., 2017), bra-ket notation can serve as an effective tool for tracking how the quantum state is transformed and affected by the interferometer's optical elements and their relational setup. This mathematical symbolic language provides a systematic way to express how each component of the setup implements a specific transformation on the quantum state of the system.

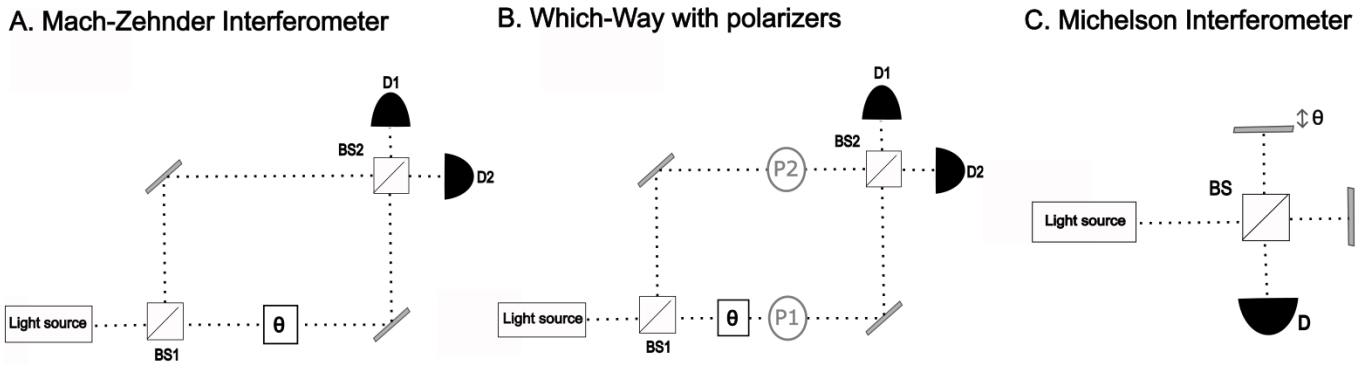
The central "meanings" that are mapped are the experimentally relevant control parameters translated to mathematical objects that preserve the probability structure of the experiment. These include the preparation of an input state by a single photon source, optical elements and their transformations on the quantum state, the spatial relations between optical components that are encoded to mathematical structures of superposition and entanglement, and the measurement outcomes represented as detection probabilities through phase relations and coherence. Together, these structures enable predictions of the measurement outcomes. Changes in experimental control parameters change the quantum state and consequently the observed detection statistics. In this way, bra-ket notation provides a symbolic description that is consistent with the quantum theory of light and serves as structural representation of empirically observed regularities. The mathematical structure captures both the statistical dependencies encoded by the Born rule and the correlations between different degrees of freedom, thus connecting the

physical configuration of the experiment to observable outcomes through quantum theory.

The most central mathematical structures of interest here are the superposition and entanglement of quantum states. Superposition is mathematically described as a linear combination of the state vectors in Hilbert space. The linear structure of Hilbert space makes the coherent addition and recombination of probability amplitudes possible. Entanglement, represented in a tensor-product Hilbert space, encodes non-separable correlations between different degrees of freedom. The Born rule, implemented through inner products in Hilbert space, provides the probabilities of measurement outcomes.

The mathematical structures described above and as they appear in the single photon interferometric experiments of interest are implemented using Dirac's bra-ket formalism. Thus, the geometry of the interferometer – how its inputs and arms, beamsplitters, phase shifters, polarisers, and detectors in the output are located – is crucial in deciding how mapping onto a mathematical bra-ket structure is done. The experimental arrangements of interest are: (A) Mach-Zehnder interferometer (MZI) with phase shifter, (B) MZI with which-way (WW) markers, and (C) Michelson interferometer. These are shown schematically in Figure 1.

Figure 1. Schematic diagrams of quantum optics experiments.



In all interferometers, a single photon light source provides a single photon state (single excitation of the field mode i.e. single-photon Fock-state) which is prepared as the input state of the interferometer. First, the photon state is affected by the first beamsplitter (BS), transforming it into a superposition of spatial modes, i.e. photon states corresponding to two different arms of the interferometer. Second, the transmitted and reflected photon states are then superposed in the second BS or again in the same BS in the Michelson interferometer setup. Finally, a single excitation is detected at either detector 1 or detector 2 in the MZI and in a single detector in the Michelson layout. The relative phase shift between the interferometer's arms is controlled with a phase shifter, often operationalised by moving one of the mirrors. Adding polarisers to the setup, now-called a which-way (WW) experiment, introduces polarisation as an additional degree of freedom relevant to the description of the system. With linear polarisers set up

orthogonally in both arms of the WW, we get an orthogonal measurement basis consisting of vertical and horizontal polarisation. These polarisation and spatial-mode degrees of freedom become entangled, resulting in a non-separable quantum state. In this study, non-separable states are referred to as entangled states. These polarisation states act as so-called marker states and can be utilised to provide the which-way information of the experiment. Using quarter wave plates instead of linear polarisers transforms the polarisation basis. The circular polarisation basis states are superpositions of the vertical and horizontal polarisation states, thereby erasing the which-way information.

The schematic diagram showing how optical components of the device are related to each other provides the basis for the mathematical (symbolic) representation of the quantum state. This is expressed using bra-ket notation. To use bra-ket notation as a mapping tool, seven rules (R1-R7) were implemented. These rules were taught as a part of the course without explicitly providing this list to students. Rules one to four (R1-R4) concerned the spatial modes of the electromagnetic field, rule five (R5) concerned the detector states, whereas rules six and seven (R6-R7) concerned the polarisation states.

R1. The transmitted spatial mode of the quantum field after the beamsplitter is denoted with $|T\rangle$ and the reflected mode with $|R\rangle$. (Transformations from two beamsplitters are $|TT\rangle,\ |TR\rangle,\ |RR\rangle,\ |RT\rangle$).

R2. The factor i is added to the reflected state to represent the fixed relative phase introduced by the beamsplitter.

R3. The field is affected by the beamsplitter, resulting in superposition of spatial modes (spatial degrees of freedom). This superposition is denoted $\frac{1}{\sqrt{2}}(|T\rangle + i\,|R\rangle)$ for a symmetric beam splitter.

R4. Adding a phase shifter to one arm of the interferometer induces a controllable relative phase factor $e^{i\theta}$ in the corresponding spatial mode.

R5. Detector states, related to the single photon detections, are denoted $|D1\rangle$ for first detector and $|D2\rangle$ for second detector.

R6. Additional linear polarisers (or polarising beamsplitters) to the system are denoted $|V\rangle$ for vertical polarisation and $|H\rangle$ for horizontal polarisation. For circular polarisation, $|L\rangle$ denotes left circularity and $|R\rangle$ denotes right circularity.

R7. Entanglement between different degrees of freedom (e.g. spatial and polarisation degrees of freedom) is denoted by a non-separable tensor product state e.g. $|V\rangle|T\rangle + |H\rangle|R\rangle$*

*There are multiple ways of writing a tensor product: $|V\rangle \otimes |T\rangle$, $|V,T\rangle$, $|VT\rangle$ (Scarani et al., 2010) but we chose $|V\rangle|T\rangle$.

Different quantum state representation schemes were used during the course to describe reflected and transmitted spatial modes. These included spatial-mode representations, such as $|T\rangle$ and $|R\rangle$ or $|\psi_U\rangle$ and $|\psi_L\rangle$, as well as occupation-number (Fock state) representations $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$. Other notation styles are of course optional and possible, but I focus mostly on the $|T\rangle$ and $|R\rangle$ style of notation as I believe it has intuitive

value for students. These rules provide simple steps on how to map experiments' physical configurations into bra-ket notation for different degrees of freedom, such as spatial modes and polarisation states. The rules relate the formal representation to the transformations of the optical components including beamsplitters, phase shifters, polarisers and detectors. Finally, students can utilize the quantum state of the system to calculate the detection probabilities using the Born rule. In terms of mathematical complexity, it is a simple calculation, and no advanced mathematical skills are required.

To implement the Born rule, only two basic rules (B1 and B2) are needed:

B1. Probability $P = |\langle\phi|\psi\rangle|^2$ meaning the projection of quantum state ϕ to the system's quantum state ψ, providing probability distribution of detections related to state ϕ.

B2. Orthogonality of the quantum states e.g. spatial modes, detector states, and polarisation states e.g. $\langle T|R\rangle = \langle D_1|D_2\rangle = \langle V|H\rangle = 0$ and $\langle T|T\rangle = \langle R|R\rangle = \langle D_1|D_1\rangle = \langle D_2|D_2\rangle = \langle V|V\rangle = \langle H|H\rangle = 1$.

Additional calculation tips for complex numbers $|z|^2 = zz^*$ and Euler identity $e^{i\theta} + e^{-i\theta} = 2\,cos(\theta)$ were provided to students. Polarisation states do not necessarily need to be orthogonal, but for simplification only orthogonal states were implemented.

First (B1), the projection with modulus squared provides a basis for calculating the measurement probabilities. Second (B2), the orthogonality of states expresses the distinguishability of photon states observed in measurement statistics. The key idea in this mapping is to give students an easy set of rules to follow when formally describing the evolution of the quantum state in various interferometric experiments. Examples of application of Dirac's bra-ket notation as a mapping and Born rule calculation can be found in the Appendix.

2.3 Mapping Meaning to Mathematics in quantum optics experiments

The knowledge base in this version of MMM consists of quantum optics experimental content knowledge, quantum physical theory, symbolic forms (mainly bra-ket) and mathematical theory (mainly basic linear algebra). There are two main epistemic forms of this e-game. First, a schematic diagram of an interferometric quantum optics experiment (see Fig. 1) is given to the students. The diagram serves as a pictorial epistemic form that encases the optical components and their spatial and causal relations, which then constitutes the symbolic (formal) operations needed. Second, the main epistemic form is mathematical formalism of quantum theory, expressed using Dirac's bra-ket notation to encode the quantum state and its evolution in the quantum optical (physical) experiment. These two epistemic forms, pictorial and symbolic, provide access to cross representational mapping, an epistemic blend that guides the inquiry. Here the abstract mathematical structure of

quantum theory using Dirac's bra-ket notation is applied to the pictorial diagram of the experiment constituted by the experimental setup in the physical domain. These two epistemic forms are navigated through a set of moves (listed below), which provide grounds for the implementation of the rules (described previously) that structure the progression in this e-game. The goal is to correctly generate the evolution of the quantum state of the system and then calculate the detection probability using the Born rule.

The moves and task expectations of quantum optics adapted MMM:

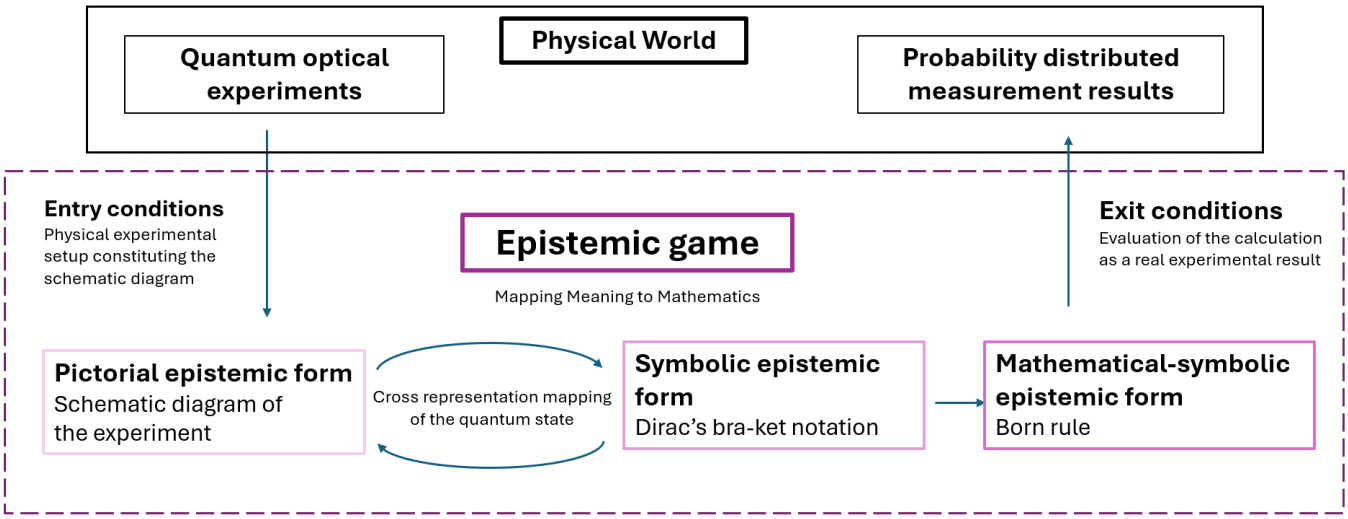
1) Develop a story: Identify the schematic illustration as a single photon quantum optics experiment with different optical components.
2) Translate: Optical components and how they affect the quantised electromagnetic field are translated into quantum states using Dirac notation (rules R1,R4,R5,R6).
3) Relate symbols: Using physical theory in accordance with the causal relations of the components depicted in the schematic illustration (rules R2,R3,R7).
4) Manipulate symbols: According to mathematical theory (rules B1,B2).
5) Evaluate and interpret: Evaluate result as quantum optics experimental outcome. Interpret the detection pattern.

The entry condition, the start of the e-game, is recognising the physical quantum optics experiment that constitutes the schematic diagram. The five moves between the entry and exit conditions listed above are described here in greater detail: the first move is to develop a story of the physical situation. After identifying the schematic drawing as an illustration of a real quantum optical experiment, the student recognizes the physical components and their relations (structure of the experiment) from the schematic illustration. In this way, the illustration serves as the first epistemic form, providing a bridge between the real experiment and the upcoming mathematical mapping. The second move is to translate physical components into mathematical symbols using Dirac's bra-ket notation. In the third move, the mathematical symbols are connected to each other according to quantum physics theory e.g. using the mapping approach to describe the physical experiment. Now the quantum state of the system is "mapped" according to the physical theory to mathematical formalism using Dirac's bra-ket notation. These first three moves are repeated until the final state of the quantum optical system is reached. In the fourth move, after the final quantum state of the system is formulated, the mathematical description is manipulated according to the Born rule using mathematical theory. The fifth move is where result of the probability calculation is evaluated as a result of a real physical experiment illustrated in the initial schematic diagram. This also serves as the exit condition, the end of the e-game, where the student's personal evaluation of the probability calculation in relation to the real physical experiment leads to the decision to continue or stop playing the e-game i.e. the answer is sufficient. Continuing the e-game means reverting to moves 1-5. The excluded moves in this e-game are understood as breaking the rules. These are identified as errors in the symbolic manipulation in the

mapping of the evolution of the quantum state or errors in the calculation of the probabilities using the Born rule. Examples of implementation of MMM are showcased in the Appendix. The main difference between Tuminaro & Redish's (2007) version of MMM and the one presented here is that instead of initially describing the physical situation, the starting point is a schematic diagram of a real physical experiment.

The MMM, as outlined above, ultimately allows students to connect the probability-based information available from the experiments to a broader conception of the underlying physical reality. This is illustrated in Figure 2. Removing the complex calculation machinery and minimising the need for mathematical manipulation, this way of implementing Dirac's bra-ket notation can serve more as a tool for reasoning, as a way of guiding the inquiry when describing the quantum optical system and the relations in it.

Figure 2. Mapping meaning to mathematics in the context of schematic-symbolic mapping when calculating detection probabilities in quantum optics experiments.



3. Methods

The data was gathered from pre-service physics teachers (N=13) attending an introductory course on quantum physics that emphasised a quantum optical approach (for a more detailed description of the course contents see (Koponen et al., 2025)). The course was offered in the 2nd or 3rd year of bachelor studies for pre-service physics teachers, after they had passed basic university physics studies. This was the first quantum physics course in their university curriculum. In physics teachers education, the course served as an introduction to the contemporary notion of the concept of the photon as a quantum state and to quantum technologies guided by quantum optics. The course contents included various quantum optics experiments that were discussed throughout the course in the lectures. These included various interferometers and double-slit experiments. Students also participated in lab work including single-photon double-slit experiment and quasi-classical quantum eraser using Mach-Zehnder interferometer with polarisers. Relevant physical

phenomena for the assignments were introduced through lectures and laboratory activities. At the beginning of the course, a set of self-study material (A1) was provided for students to practice and refresh their knowledge on central physical and mathematical skills e.g. basic linear algebra, complex numbers, Euler's formula, and classical polarisation. The data evaluated consisted of two assignments from the course exam and eight assignments from weekly exercises, all part of the course grade evaluation. These assignments were constructed and motivated in a similar manner described by the mapping approach introduced in section 2. Therefore, MMM provides obvious viewpoint to approach these assignments. Students were expected to map schematic representations of quantum-optical experiments onto symbolic descriptions using Dirac's bra-ket notation and subsequently apply the Born rule to calculate detection probabilities. The exam included mapping and a Born rule assignment applied to a Mach-Zehnder interferometer. The exam was done as a paper exam and conducted under constant supervision. The weekly assignments included mapping and Born rule assignments in the context of Mach-Zehnder interferometer (A2), MZI extended to which-way experiments using linear and circular polarisation (A3), and Michelson interferometer (A5). For schematic diagrams of the experiments see Fig. 1 and examples of the mapping and Born rule in the Appendix. The resulting assignment responses formed the dataset analysed in this study. A total of 55 mapping assignments and 50 Born rule assignments were returned and analysed. Assignments were part of the course grading but not mandatory. This resulted in some participants not returning all the course assignments. The course structure is described in Table 1.

Table 1. Course structure. L = Lecture, A = Assignment. A4 marked with asterisk did not include mapping or Born rule assignments.

Week	Activities	Week	Activities
1	L1, L2, A1	**5**	L7, L8
2	A2, L3	**6**	L9, A4*
3	L4, L5	**7**	A5
4	A3, L6	**8**	Exam

The students' level of performance for each assignment was evaluated using five level rubrics (see Table 2 and 3). Only submitted assignments containing work were evaluated. From these evaluation results, the mean value of the performance level for each assignment was calculated together with percentage distribution across rubric levels (see Table 6 and Table 8). The representational schemes used by participants during bra-ket mapping (see Figure 3 and 4), as well as their approach to solving the Born rule (see Table 9), were documented and evaluated in relation to students' previously acquired performance level. If a student did not attain the highest performance level on an assignment, the underlying reason was identified and coded as an error (see Table 7 and 10). The study conducted served as a pilot for the use of the rubrics presented below.

Examples of the students' answers (anchor responses) that correspond to a given level in the rubrics in Tables 2 and 3 are provided in section 4 following.

Table 2. Rubric for evaluating the performance level in Dirac's bra-ket mapping assignments.

Level	Description
0 – Wrong	Totally wrong mapping.
1 – Minimal	An attempt at mapping or some kind of qualitative description in the right direction.
2 – Partially correct	Correct mapping including first beamsplitter.
3 - Mostly correct	Correct mapping including second beamsplitter with minor mistake.
4 - Correct	Correct mapping and final formulation of the system's quantum state.

Table 3. Rubric for evaluating the performance level in Born rule assignments.

Level	Description
0 – Wrong	Wrong equation for Born rule and wrong calculation.
1 - Minimal	Correct equation for Born rule with wrong calculation or a qualitative description.
2 - Partially correct	Correct equation for Born rule, orthogonality of the states applied, attempt at Conjugate rule or Euler's formula but wrong calculation.
3 - Mostly correct	Correct equation for Born rule, orthogonality of the states applied, use of Conjugate rule or Euler's formula with minor calculation error or confusion with the final result.
4 - Correct	Correct equation for Born rule, orthogonality of the states applied, correct calculation of Conjugate rule or Euler's formula, correct result.

The explicit nature of the formalism, combined with the rubrics provided in Table 2 and 3 together with anchor responses (Table 4), enables the evaluation of students' performance levels from the data and thereby provides a means to answer Research questions 1 and 3. The errors detected during the evaluation process formed the basis for answering Research question 2. Error categories were developed inductively from the data set. First, all student responses were reviewed to identify recurring errors. Based on these observations, a codebook of error categories was generated (see Table 7 and Table 10). Finally, the codebook was applied to the entire data set. Multiple errors could occur within a single response. Representation schemes were analysed in a similar fashion. First, recurring representation schemes were identified inductively from the data, categorised, and subsequently analysed across the entire data set. Multiple representation schemes could occur within a single response.

4. Results

4.1 Examples of anchor responses for rubric levels

Anchor responses illustrating performance levels 2-4, described in Tables 2 and 3, are presented in detail below. Examples of the anchor responses, as they appeared in students' answers (in the exam task about MZI), are presented in Table 4 for bra-ket mapping and in Table 5 for Born rule calculations.

Table 4. Examples of Anchor responses of Dirac's bra-ket mapping. L refers to rubric level of Table 2. Mathematical expressions used by students and the rubric level explanations are presented.

L	Expression	Explanation
2	$\|\psi\rangle = \|1\rangle_{inp}\|0\rangle_{inp} \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\|1\rangle\|0\rangle + i\|0\rangle\|1\rangle) \xrightarrow{e^{i\theta}} \frac{1}{\sqrt{2}}(e^{i\theta}\|1\rangle\|0\rangle + i\|0\rangle\|1\rangle)$ Followed by statement, *"BS2 does not alter the quantum state"*.	Mapping done correctly including BS1. Failed to map the second beamsplitter BS2.
3	$\|\psi\rangle = \|1\rangle_{inp}\|0\rangle_{inp} \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\|T\rangle + \mathrm{i}\,\|R\rangle) = \frac{1}{\sqrt{2}}(\|0\rangle_R\|1\rangle_T + \|1\rangle_R\|0\rangle_T)$ $\rightarrow \frac{1}{\sqrt{2}}(e^{i\theta}\|T\rangle + \mathrm{i}\,\|R\rangle)$. $D1 = \|TT\rangle + \|RR\rangle,\ D2 = \|TR\rangle + \|RT\rangle$ $\xrightarrow{BS2} \frac{1}{\sqrt{4}}(e^{i\theta}(\|TT\rangle + i\|TR\rangle) + i(\|RT\rangle + i\|RR\rangle)$. Followed by statement, *"I am unable to draw a conclusion"*.	Maps correctly up to BS2, recognises states related to detectors, but fails to provide the final results.
4	Starting with explanation, "*a light source creates a photon, which is described with* $|1\rangle$ *and the vacuum state with* $|0\rangle$". $\|\psi\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\|T\rangle + \mathrm{i}\,\|R\rangle) \xrightarrow{e^{i\theta}} \frac{1}{\sqrt{2}}(e^{i\theta}\|T\rangle + \mathrm{i}\,\|R\rangle)$ $\|T\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(\|D_1\rangle + i\|D_2\rangle)$, $\|R\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(\|D_2\rangle + i\|D_1\rangle)$ $\|\psi\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(e^{i\theta}\frac{1}{\sqrt{2}}(\|D_1\rangle + i\|D_2\rangle) + i\frac{1}{\sqrt{2}}(\|D_2\rangle + i\|D_1\rangle)$ $= \frac{1}{2}((e^{i\theta} - 1)\|D_1\rangle + i(e^{i\theta} + 1)\|D_2\rangle)$	Performs correct mapping with correct final formulation of the quantum state.

Table 5. Examples of anchor responses of Born rule calculation. L refers to rubric level of Table 3. Mathematical expressions used by students and the rubric level explanations are presented.

L	Expression	Explanation
2	$P_{D1} = \|\langle D_1\|\psi\rangle\|^2 = \left\|\langle D_1\|\frac{1}{\sqrt{2}}((e^{i\theta}-1)\|D_1\rangle + (e^{i\theta}+1)i\|D_2\rangle)\right\|^2$ $P_{D1} = \left\|\frac{1}{\sqrt{2}}(e^{i\theta}-1)\right\|^2 = \frac{(e^{i\theta}-1)^2}{2} = \frac{(cos\theta + i\,sin\theta - 1)^2}{2}$ $= \frac{(cos\theta^2 - sin\theta^2 + 2isin\theta cos\theta - cos\theta - i\,sin\theta + 1)}{2}$	Correct equation for Born rule, orthogonality of the states applied but failing in the attempt to use Euler's formula.
3	$P_{D1} = \|\langle D_1\|\psi\rangle\|^2 = \left\|\langle D_1\|D_1\rangle\frac{(e^{i\theta}-1)}{2} + \langle D_1\|D_2\rangle\frac{(e^{i\theta}+1)}{2}\right\|^2$ $\langle D_1\|D_1\rangle = 1$ and $\langle D_1\|D_2\rangle = 0$ $= \left\|\frac{e^{i\theta}-1}{2}\right\|^2 = \left\|\frac{(cos\theta + i\,sin\theta - 1)}{2}\right\|^2 = \frac{(cos\theta + i\,sin\theta - 1)(cos\theta - i\,sin\theta - 1)}{2}$ $= \frac{cos\theta^2 + sin\theta^2 - 2cos\theta + 1}{2} = \frac{1 + 1 - 2cos\theta}{2} = 1 - cos\theta$	Correct equation for Born rule, orthogonality of the states applied, use of Euler's formula with minor calculation error as the denominator is not raised to the power of two. Nearly correct final result.
4	$P_{D1} = \|\langle D_1\|\psi\rangle\|^2$, $\langle D_1\|\psi\rangle = \langle D_1\|\frac{1}{2}(e^{i\theta}-1)\|D_1\rangle + \langle D_1\|\frac{1}{2}i(e^{i\theta}+1)\|D_2\rangle$ $\langle D_1\|D_1\rangle = 1$ and $\langle D_1\|D_2\rangle = 0$ so $\langle D_1\|\psi\rangle = \frac{1}{2}(e^{i\theta}-1)$ $P_{D1} = \left\|\frac{1}{2}(e^{i\theta}-1)\right\|^2 = \frac{1}{4}(e^{i\theta}-1)(e^{-i\theta}-1)$ $= \frac{1}{4}(cos\theta + i\,sin\theta - 1)(cos\theta - i\,sin\theta - 1) = \frac{1}{4}(cos\theta^2 + sin\theta^2 - 2cos\theta + 1)$ $= \frac{1}{4}(2 - 2cos\theta) = \frac{1}{2}(1 - cos\theta)$	As in level 3, but correct use of Euler's formula and correct final result.

4.2 Dirac's bra-ket mapping

The results of the students' performance level in the bra-ket mapping of the quantum state are presented in Table 6.

Table 6. The mean performance level and percentage distribution across rubric levels in each mapping assignment. N represents the number of assignments analysed and the number of answers in each rubric level are presented in parentheses. Maximum performance level is 4.

Exercise	N	Mean	0	1	2	3	4
Exam (MZI)	**13**	**3.15**	0 %	0 %	15.4 % (2)	53.8 % (7)	30.8 % (4)
MZI	**12**	**3.92**	0 %	0 %	0 %	8.3 % (1)	91.7 % (11)
WW: Linear	**11**	**3.82**	0 %	0 %	0 %	18.2 % (2)	81.8 % (9)
WW: Circular	**11**	**3.73**	0 %	0 %	0 %	27.3 % (3)	72.7 % (8)
Michelson	**8**	**3.13**	0 %	12.5 % (1)	12.5 % (1)	25 % (2)	50 % (4)

From the percentage distributions (see Table 6), we can see that the students performed well in all mapping assignments as the majority of responses were classified at level 3 or 4. Performance level in exam scored a relatively high mean value of 3.15 while 84.6% of answers were classified as level 3 or 4 (see Table 6). The performance level in the exam assignments reflects students' independent ability to apply the mapping, since the exam was carried out in a controlled setting. Throughout the course, in all of the weekly assignments, MZI and even the more complex experiments e.g. Which-way experiments with linear and circular polarisers, the level of performance remained very high, with a mean value varying between 3.73 and 3.92. These were the highest scores and can be expected because the students could use course materials and peer support while performing the assignments. The Michelson interferometer assignment scored the lowest mean value of 3.13 with only 8 participants returning the task. Two of these answers (25%) received lowest response classifications and were classified in levels 1 or 2. This might reflect transfer difficulty as the Michelson interferometer was not covered in the course lectures or materials and was a new context in which students needed to apply their skills. An additional factor might be the geometrical setup of the Michelson interferometer. Compared to the MZI, the spatial modes in the Michelson interferometer are less explicit, and a single beamsplitter transforms the state twice; also, the measured and unmeasured output modes require additional interpretation. From these results, the overall performance level can be perceived as being quite high. This result suggests that applying Dirac's bra-ket mapping of the quantum state to different quantum optics interferometric experiments was not too demanding for the participants.

The errors detected in the students' answers, examined in relation to the mapping rules (see section 2.2), are presented in Table 7.

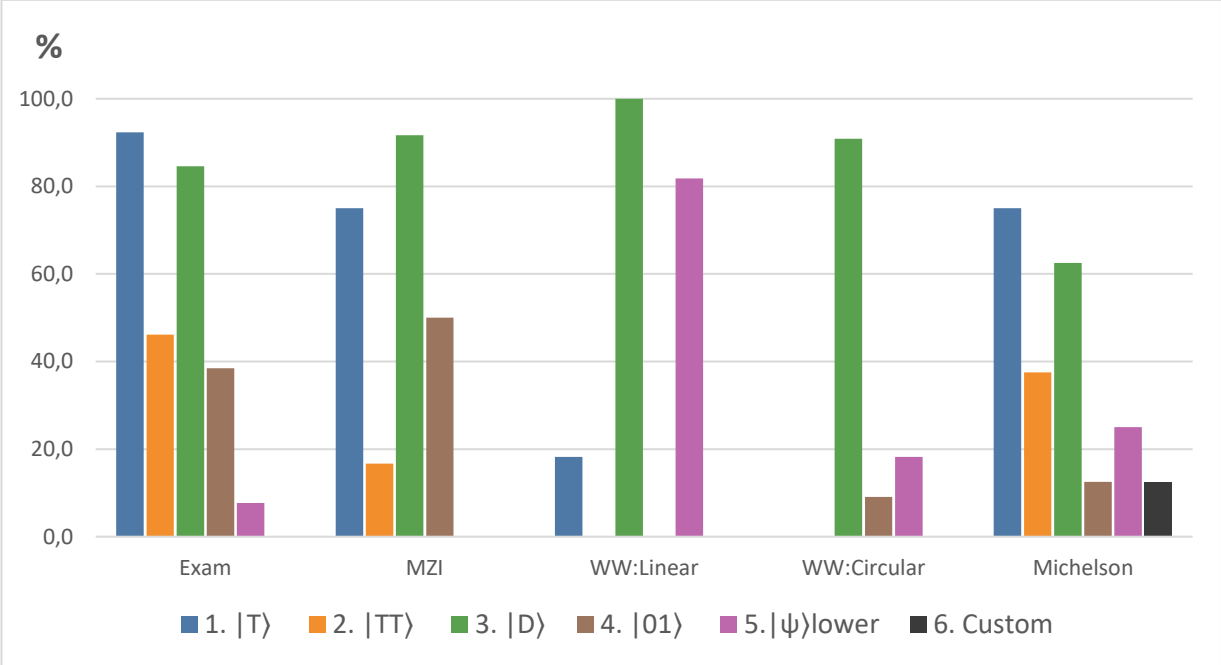
Table 7. Coding, number, and description of errors detected in the mapping assignments.

Error:	N All	N Exam	Description of the error
E1a	1	0	After BS1, difficulties in correctly relating i to the reflected state.
E1b	8	2	After BS2, difficulties in correctly relating i to the reflected state.
E2	3	3	Detector states $\|D\rangle$ missing from the mapping.
E3	1	1	Issues in the evolution from $\|\psi\rangle$ to the superposition $\|\psi\rangle_{\text{upper}} + \|\psi\rangle_{\text{lower}}$.
E4	3	1	Final formulation of the mapping lacks clarity.
E5	32	2	The single-photon state is not properly addressed using Fock states, either as the input state or in relation to the $\|T\rangle$ and $\|R\rangle$ states.
E6	1	1	Incorrect claim that BS2 has no effect on the quantum state.
E7	3	1	A minor calculation error is present.
EQE1	1	0	An incorrect equation is used for circular polarisation.

The most important thing to note is that the mathematical structures of superposition (rule R3), phase-shift (rule R4), and entanglement (rule R7) were applied well as only one error (E3) was recorded. The absence of representations of the input state or the transmitted and reflected state using Fock states (error E5) were the most usual errors (see Table 7). In the context of MZI it was applied well, but not in the context of which-way assignments with linear and circular polarisation or with Michelson interferometer. Students might face some confusion about the occupation-number representation of the input state (i.e. single photon state as Fock state) or perceive it as unnecessary. However, interestingly, single photon states as Fock states were correctly presented in the final exam. The second most common error was applying i to the reflected state (rule R2) for the second beamsplitter (error E1b) or even thinking that the second beamsplitter would not affect the quantum state (E6). Applying i to the reflected state after first beamsplitter was known well. Connection of the spatial states to the detector states and confusion about the final formulation of the quantum state caused some challenges (errors E2 and E4). This is connected to the application of rule R5 (detector states). Together, these notions indicate that it was difficult for only a few students to produce the correct form or even understand the goal of the mapping. The rest of the errors were minor calculation errors.

The students used multiple representation schemes when answering the course and exam assignments (see Figure 3).

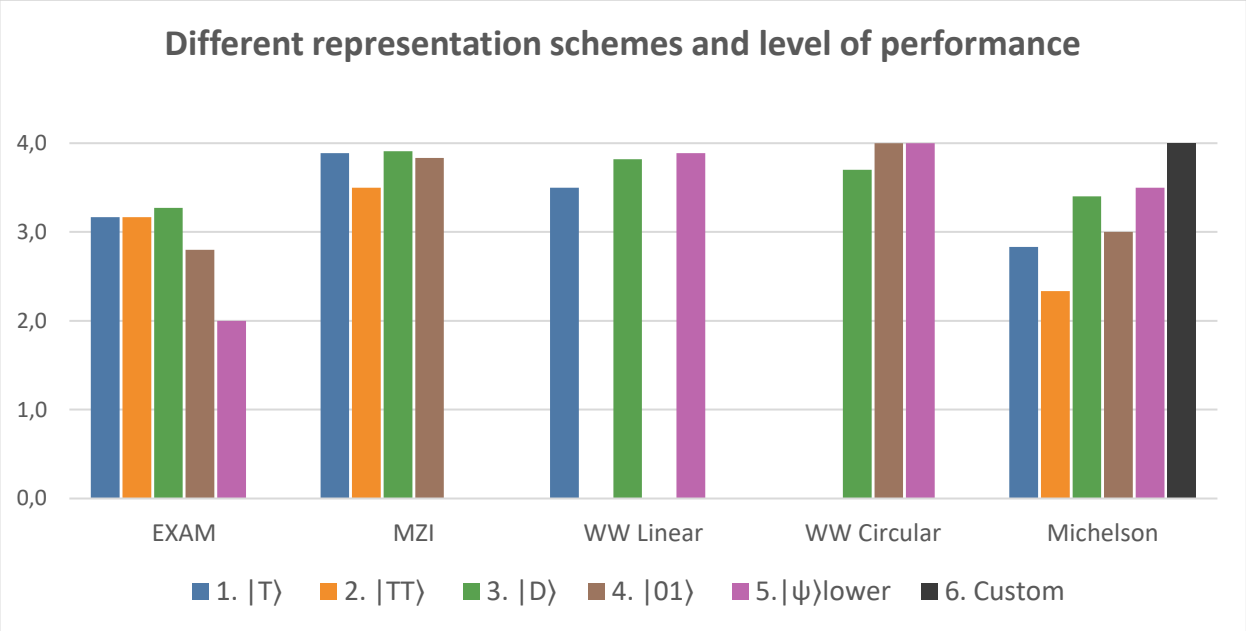
Figure 3. The percentage of different representation schemes used when solving each assignment. The representation schemes are not mutually exclusive.



The results in Fig. 3 show that in most cases students relied on the representational scheme $|T\rangle$ for the transmitted and $|R\rangle$ for the reflected state: about 80% used this notation in the context of Mach-Zehnder interferometer and Michelson interferometer. Also, the representation scheme that encased transmitted and reflected state as $|\psi\rangle_{\text{upper}}$ and $|\psi\rangle_{\text{lower}}$ was popular (80% of the answers in WW: Linear). This is an expected result since these notations were used in the course materials. Interestingly, with the Michelson interferometer, which was not presented in the course, the students relied mostly on $|T\rangle$ and $|R\rangle$ notation, indicating that it might be the most intuitive method for them. Addressing the transmitted and reflected state after BS2 using $|TT\rangle$ and $|RR\rangle$ notation was seldom used in the first course assignments on the Mach-Zehnder interferometer, but almost 40% of the students adopted it in the Michelson interferometer assignment and well over 40% in the MZI context in the exam. Throughout the course, identifying the detector states with $|D\rangle$ notation was present in 80-100% of the answers. For the Michelson interferometer, some students made up their own "custom" notation for the undetected output state after BS2. Some of the students adopted the Fock state representation, e.g. $|01\rangle$ and $|10\rangle$, and mapped the entire quantum state evolution using it.

The mean value of the level of performance was calculated for each notation used. If notation was used at any time when solving the assignment, the performance level was then included in the mean calculation. These are shown in Figure 4.

Figure 4. The level of performance using different representational schemes visualized. The representation schemes are not mutually exclusive.



The results in Fig. 4 suggest no apparent difference in the level of performance in the notations used with the first three course exercises MZI, WW Linear and Circular. In the exam in the context of MZI, the most well-managed was $|T\rangle$, also with the extension of $|TT\rangle$, and Fock state $|01\rangle$ notation; $|\psi\rangle_{\text{upper}}$ scored a clearly lower level of performance, but very few students adopted this in the exam context so the difference should be interpreted with caution. Fig. 4 shows some variation in the performance level in the context of Michelson interferometer. There is no clear reason for the difference, but given the small number of answers with $|TT\rangle$ or custom notation, the results should be interpreted with caution. Fig. 3 and 4 show that across the most common representational schemes adopted by the students in each assignment, bra-ket notation was used with a consistently high level of performance. Considering the variation in representational schemes employed across different assignments and quantum optics experimental layouts, no clear relationship between notation style and performance level was observed. Below are representative student examples of using Dirac's bra-ket notation as mapping with different representational schemes.

Example 1: Student 4 in exam using Fock state $|01\rangle$ notation:

> *At the beginning the state is* $|\psi\rangle = |0\rangle|1\rangle$. *After the beamsplitter state is a superposition of transmitted and reflected states. Reflected state acquires a half-wavelength phase shift and* $e^{i\pi} = i$.
>
> $|0\rangle|1\rangle \xrightarrow{\text{BS1}} \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + \text{i}|1\rangle|0\rangle)$ $\quad normalized\ \frac{1}{\sqrt{2}}$

Effects of the mirrors on reflected and transmitted are the same so let's not complicate the analysis with adding their phases. Transmitted state is affected by phase shift so
$\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + i|1\rangle|0\rangle) \xrightarrow{e^{i\theta}} \frac{1}{\sqrt{2}}(e^{i\theta}|0\rangle|1\rangle + i|1\rangle|0\rangle)$

Beamsplitter BS2 interacts similarly as before so
$|0\rangle|1\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + i|1\rangle|0\rangle)$ and $|1\rangle|0\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(|1\rangle|0\rangle + i|0\rangle|1\rangle)$

By combining this information, we obtain
$\frac{1}{\sqrt{2}}(e^{i\theta}|0\rangle|1\rangle + i|1\rangle|0\rangle) \xrightarrow{BS2} \frac{1}{2}[e^{i\theta}(|0\rangle|1\rangle + i|1\rangle|0\rangle) + i(|1\rangle|0\rangle + i|0\rangle|1\rangle)]$
$= \frac{1}{2}(e^{i\theta}|0\rangle|1\rangle + ie^{i\theta}|1\rangle|0\rangle + i|1\rangle|0\rangle - |0\rangle|1\rangle)$
$= \frac{1}{2}(e^{i\theta} - 1)|0\rangle|1\rangle + i(e^{i\theta} + 1)|1\rangle|0\rangle$

Example 2: Student 5 in exam using $|T\rangle$ and $|TT\rangle$ notation.

Before BS1 photon state is $|\psi\rangle = |0\rangle|1\rangle$
$|\psi\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}(|T\rangle + i|R\rangle)$
$|\psi\rangle \xrightarrow{\theta} \frac{1}{\sqrt{2}}(e^{i\theta}|T\rangle + i|R\rangle)$

Let's investigate quickly $|D_1\rangle$ *and* $|D_2\rangle$ *states.*
$|D_1\rangle = |T\rangle|T\rangle + i|R\rangle|R\rangle$
$|D_2\rangle = |T\rangle|R\rangle + i|R\rangle|T\rangle$
$|\psi\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(e^{i\theta}|T\rangle|T\rangle + ie^{i\theta}|T\rangle|R\rangle) + i(\frac{1}{\sqrt{2}}(|R\rangle|T\rangle + i|R\rangle|R\rangle))$
$= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(e^{i\theta}|TT\rangle - |RR\rangle + i(e^{i\theta}|TR\rangle + |RT\rangle)))$
$= \frac{e^{i\theta}-1}{2}|D_1\rangle + i\frac{e^{i\theta}+1}{2}|D_2\rangle$

Example 3: Student 9 in WW Linear assignment using $|\psi_L\rangle$ notation:

Light travelled through P1 is horizontally polarised (let's mark this state as $|H\rangle$*) and light travelled through P2 is vertically polarised (let's mark this state as* $|V\rangle$*).*

After beamsplitter BS1 the quantum state is $\frac{1}{\sqrt{2}}(|\psi_U\rangle + |\psi_L\rangle)$

After phase shifter the quantum state is $\frac{1}{\sqrt{2}}(|\psi_U\rangle + e^{i\theta}|\psi_L\rangle)$

After polarizers the quantum state is $\frac{1}{\sqrt{2}}(|\psi_U\rangle|H\rangle + e^{i\theta}|\psi_L\rangle|V\rangle)$

After second beamsplitter the quantum state is
$\frac{1}{2}((i|D_1\rangle + |D_2\rangle)|H\rangle + e^{i\theta}(|D_1\rangle + i|D_2\rangle)|V\rangle)$

Reorganising the terms so the quantum states is
Student reorganizes terms until arriving at the final form of the quantum state
$= \frac{1}{2}((i|H\rangle + e^{i\theta}|V\rangle)|D_1\rangle + (|H\rangle + ie^{i\theta}|V\rangle)|D_2\rangle)$

These answer examples demonstrate how students' connected mathematical structures to the components in quantum optics experiments using different representation schemes.

4.3 Born rule

The results of the students' mean performance level together with percentage distributions across rubric levels in the Born rule calculations are presented in Table 8. The analysis reveals that students had two different ways of solving the Born rule calculation, using either the conjugate rule or Euler's formula (see Table 9).

Table 8. The mean performance level and percentage distribution across rubric levels in each Born rule assignment. N represents the number of assignments analysed and the number of answers in each rubric level are presented in parentheses. Maximum level was 4 in all assignments.

Exercise	N	Mean	0	1	2	3	4
Exam (MZI)	**12**	**2.92**	0 %	25 % (3)	8.3 % (1)	16.7 % (2)	50 % (6)
MZI	**11**	**3.91**	0 %	0 %	0 %	9.1 % (1)	90.9 % (10)
WW: Linear	**12**	**3.42**	8.3 % (1)	0 %	0 %	25 % (3)	66.7 % (8)
WW: Circular	**11**	**3.45**	9.1 % (1)	0 %	0 %	18.2 % (2)	72.7 % (8)
Michelson	**4**	**3.25**	0 %	0 %	25 % (1)	25 % (1)	50 % (2)

From the weekly assignments, the Michelson interferometer assignment scored the lowest mean value of 3.25 and levels ranging from 2 to 4, though only 4 participants returned the task. The MZI scored the highest mean of 3.91. While the which-way experiments with linear polarisation and circular polarisation involved more complex situations, the mean level of performance of the participants remained high while the distributions were more polarised, as some answers were classified as level 0 and others as levels 3 or 4. Overall, the results show a good level of performance when applying the Born rule in the context of various interferometric quantum optics experiments. In the exam, most of the students performed well, as 66.7% of the assignments analysed were classified as level 3 or 4. In the exam 25% performed poorly, reaching only level 1. One student left answering the Born rule assignment totally blank in the exam.

Table 9. Distribution and level of performance in using different solution methods.

Method	Mean level	N	%
Conjugate rule	3.72	29	58%
Euler's formula	3.71	14	28%
Not shown	1.43	7	14%

Table 9 shows that when applying the Born rule, the conjugate rule was the preferred mathematical method, and the second popular method was Euler's formula. These intermediate calculations were not presented in the rest of the cases. The results in Table 9 show that the performance level in using the conjugate rule and Euler's formula were similarly high. These methods together cover 85% of the answers, and thus the results indicate that the students had sufficient mathematical knowledge for the mathematical tasks required in these assignments. The cases where students did not provide details of the intermediate steps of their calculations or reasoning resulted in significantly lower levels of performance. These results suggest that applying the conjugate rule and Euler's formula did not pose substantial difficulties for the students. Combined with the performance levels reported in Table 8, the findings indicate that the mathematical skills required to apply the Born rule are within the capabilities of pre-service physics teachers.

Table 10. Number and description of errors in Born rule assignments

ERROR	N	Description of error (misstep)
E1	6	Minor calculation errors are present.
E2	6	Difficulties in calculating the inner product.
E3	6	Difficulties in calculating the modulus squared.
E4	5	Incorrect quantum state is used in the projection.

The errors made in Born rule assignments are distributed evenly, thus no specific errors occurred more frequently than any other (see Table 10). Compared to the number of Born rule assignments analysed (N=50), the number of errors was low, suggesting that applying the Born rule in a quantum optics experimental context was a suitable assignment for pre-service physics teachers.

Table 11. Distribution of different ways of representing orthogonality of the states when solving the Born rule.

Method	Percentage
Mathematical	27%
Mathematical & Qualitative	15%
Qualitative	8%
Not stated	50%

The results presented in Table 11 show that the orthogonality of the states was not mentioned at all in 50% of the answers. The orthogonality of the states was expressed using only mathematical representation in 27% of all answers, using both mathematical and qualitative description in 15% and as a qualitative notion in only 8%. This result might reflect on conceptual issues in understanding the meaning of orthogonality of the states in the experimental situation or just be a notational shorthand when performing the calculation.

4.4 Limitations

The study has limitations that need to be taken into account. First, the rubrics in Table 2 and 3 were decided by the author alone as well as all the coding based on those rubrics. Consequently, no inter-rater agreement could have been possible. Second, using course assignments in the form of weekly exercises as research data has its limitations, because it is impossible to know what kind of co-operation or external support the students used while performing the weekly assignments. However, the data gathered as part of the exam was well-controlled and reflects the ability of a single student. Third, the data was gathered from a single course focusing on quantum optics, limiting the generalisability of the findings to other quantum physics contexts. The results of this study should therefore be interpreted as exploratory and future research would require replication with larger student populations.

5. Discussion and Conclusion

In this study, I conducted exploratory research on pre-service physics teachers' quantum state mapping using bra-ket notation and Born rule calculation in the context of interferometric quantum optics experiments. The research data consisted of assignments from an introductory course on quantum physics that emphasised a quantum optical approach. The design of the assignments parallels the viewpoint of the epistemic game of mapping meaning to mathematics (Tuminaro & Redish, 2007) but contextualised as a schematic-symbolic mapping of quantum optical experiments. The central aim of this approach was to minimise the mathematical skills required for learning Dirac's bra-ket notation and

emphasise the connections between mathematical structures and the meanings that they encode in the context of quantum optics experiments. In these assignments, the focus is on using formalism as a symbolic mapping tool rather than a technical calculating tool. The schematic diagram provides students with concrete grounds to describe quantum optical experiments using formal descriptions. This serves as a bridge between formalism and natural language description of concepts such as superposition and entanglement.

The study was guided by three research questions. The first research question is answered by the results, which reveal the students' engagement with bra-ket notation in a cross-representational mapping of the quantum state evolution from a schematic diagram of quantum optics experiment and in the application of the Born rule. Students mostly used the mathematical structures of superposition, entanglement, and phase-shifts of the quantum states correctly in describing the interferometric experiments. The results suggest that bra-ket formalism implemented as a mapping together with the Born rule might be suited for introductory level teaching for pre-service physics teachers and might serve as a starting point for learning the mathematical structures that are central in quantum theory of light.

In an extensive literature review, Bouchée et al. (2022) described students having difficulties in connecting mathematical formalism to the observed quantum phenomena. Difficulties involving state vectors, operators, observables, and eigenstates as well as other formal applications of Dirac notation are recorded in many studies (Lautesse et al., 2026; Marshman & Singh, 2018; Singh & Marshman, 2013). The mapping approach introduced here and contextualised in real quantum optics experiments could provide a way to introduce and construct the "meaning" behind these mathematical structures and concepts. Similarly to Levy & Singh (2025), the goal is not to dismiss the more formal approach but rather provide a basis to build on for more advanced mathematical treatments of quantum mechanics. The benefit of quantum optics contextualisation for physics teacher education is that many contemporary upper-secondary level approaches to quantum physics emphasise quantum optics and introduce two-state systems with interferometers (Bitzenbauer, 2021; Bitzenbauer & Meyn, 2021; Hennig, Tóth, Förster, et al., 2024; Zuccarini et al., 2024). Therefore, preparing pre-service physics teachers in quantum optics is recommended.

The second research question is answered by examining the errors that students make in the mapping of the quantum state. In this study, the most common errors concerned the omission of Fock state representation of the input state. Whether this reflects the students' lack of understanding or simply that students perceive it as unnecessary requires further investigation. Overall, students performed well in the application of the Born rule, with only occasional technical mathematical errors. Instead, a common problem was failure to address the orthogonality of the quantum states. This might indicate a lack of understanding of the meaning of the orthogonality of the quantum states or be the result of a notational shortcut that might be obvious or easy to derive for students. More research is needed to clarify the interpretation of this result. Orthogonality is a conceptual backbone that allows students to make correct inferences about what can and cannot

interfere or can be distinguished. Therefore, in improving teaching, it still might be worth the while to consider placing a greater emphasis on the meaning of the orthogonality of the quantum states and the meaning of the Fock state representation of single photon input states.

The third research question is answered by the result of the analysis of students' use of different representational schemes when using bra-ket notation. Variation was recorded throughout the course in different assignments, often following the style used in the lectures or course material regarding a particular experiment. With most of the assignments, the different representation schemes did not result in major differences in the level of performance. In the context of this research, these results suggest that the simple guiding "rules" provided to map the quantum state from a schematic diagram of the quantum optics experiment (pictorial epistemic form) to Dirac's bra-ket notation (symbolic epistemic form) were enough for introductory-level application and not bound to the specific symbols used, provided that the symbols were intuitive enough to be connected to the different sections and components of the experiment. One possible interpretation of this result can be drawn from Gire & Price (2015), who identify *high degree of individuation* and *symbolic support* as important characteristics of Dirac notation when manipulating probability amplitudes and quantum states. Despite the different representation schemes the underlying structure of Dirac notation was retained as quantum states continued to be represented as kets. This preservation of individuation and asymmetry may support the application of operational rules (Gire & Price, 2015). The results show pre-service physics teachers' flexibility in using different representation schemes and might indicate an ability to operate with different representation schemes suitable for their future secondary-level teaching context when using reduced Dirac formalism. The results presented parallel the findings of Gire & Price (2015) that Dirac notation seems appropriate for supporting undergraduate reasoning of quantum systems.

Overall, these exploratory results suggest that the mapping approach can support introductory-level engagement with bra-ket notation and the Born rule in the context of quantum optics experiments.

Research ethics

Artificial intelligence

For the purposes of language editing, minor use of Copilot was implemented. The author critically evaluated all the AI suggestions and is responsible for all that is written.

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Informed consent statement

The research adhered to the ethical standards of the University of Helsinki and complied with the guidelines established by the Finnish National Board on Research Integrity. Participation was entirely voluntary. Informed consent was obtained from all research participants. Participants received sufficient information about the study, and their anonymity was safeguarded throughout the research process.

Data availability statement

Data generated and analysed during this study is available from the author on request.

Conflicts of Interest

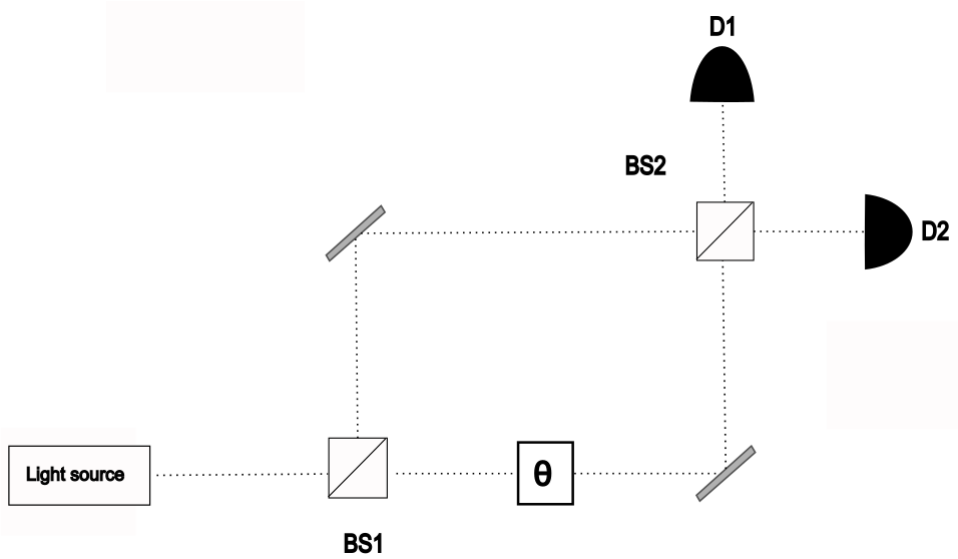
The author declares no conflicts of interest.

Appendix. Examples of the mapping approach and Born rule applications in quantum optics experiments

Spatial modes after BS1 are represented as $|T\rangle$ and $|R\rangle$ and after BS2 $|TT\rangle$, $|TR\rangle$, $|RR\rangle$ and $|RT\rangle$ and describe the spatial-mode degree of freedom. Polarisation states are represented as $|H\rangle$ and $|V\rangle$ and describe the polarisation degree of freedom. Fock states (occupation-number eigenstates) are represented as $|1\rangle$ and $|0\rangle$, while two-mode occupation-number states are represented as $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$.

Mach-Zehnder interferometer

Figure 5. Schematic illustration of the Mach-Zehnder interferometer.



$$|\psi\rangle = |1\rangle_{\text{inp}}|0\rangle_{\text{inp}} \tag{1}$$

where $|1\rangle_{\text{inp}}$ is the single photon input state and $|0\rangle_{\text{inp}}$ is the vacuum state (zero photon state). The first beamsplitter (BS1) transforms the input state according to rules R1, R2 and R3:

$$\xrightarrow{BS1} \frac{1}{\sqrt{2}}(|T\rangle + i|R\rangle) \tag{2}$$

The phase shifter in the lower arm affects the transmitted state according to rule R4:

$$\xrightarrow{e^{i\theta}} \frac{1}{\sqrt{2}}(e^{i\theta}|T\rangle + i|R\rangle) \tag{3}$$

After a second beamsplitter (BS2) applying rules R1, R2, and R3 again to get the final transformation of the quantum state:

$$\xrightarrow{BS2} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{i\theta}(|TT\rangle + i|TR\rangle) + \frac{1}{\sqrt{2}}i(|RT\rangle + i|RR\rangle)\right) \tag{4}$$

Identifying spatial modes related to the two detectors according to rule R5, we get the final form:

$$|\psi\rangle = \frac{e^{i\theta}-1}{2}\,|D_1\rangle + \frac{e^{i\theta}+1}{2}\,i|D_2\rangle \tag{5}$$

Optionally this can be done by using a Fock state-based notation style:

$$|\psi\rangle = |1\rangle_{inp}|0\rangle_{inp} \xrightarrow{BS1} \frac{1}{\sqrt{2}}(|0\rangle_R|1\rangle_T + i\,|1\rangle_R|0\rangle_T) \tag{6}$$

$$\xrightarrow{e^{i\theta}BS2} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{i\theta}(|1\rangle_{D1}|0\rangle_{D2}+i|0\rangle_{D1}|1\rangle_{D2})+\frac{1}{\sqrt{2}}i(|0\rangle_{D1}|1\rangle_{D2}+i|1\rangle_{D1}|0\rangle_{D2})\right) \tag{7}$$

$$= \frac{e^{i\theta}-1}{2}\,|1\rangle_{D1}|0\rangle_{D2} + \frac{e^{i\theta}+1}{2}i|0\rangle_{D1}|1\rangle_{D2} \tag{8}$$

Where $|1\rangle_T$ and $|0\rangle_T$ correspond to the single photon state and vacuum state related to transmitted spatial state respectively, and similarly $|1\rangle_R$ and $|0\rangle_R$ to the single photon state and vacuum state related to the reflected spatial state. This representation clearly depicts the single-photon entangled state.

We have the quantum state of the Mach-Zehnder interferometer from Eq. 5. Next, we implement the Born rule and calculate the probabilities of detection in each detector (rule B1).

$$P_{D1} = |\langle D_1|\psi\rangle|^2 = \left|\langle D_1|\left(\frac{e^{i\theta}-1}{2}\,|D_1\rangle + i\frac{e^{i\theta}+1}{2}|D_2\rangle\right)\right|^2 \tag{9}$$

$$= \left|\frac{e^{i\theta}-1}{2}\langle D_1|D_1\rangle + i\frac{e^{i\theta}+1}{2}\langle D_1|D_2\rangle\right|^2 \tag{10}$$

For the inner product, we implement rule B2, the orthogonality of states:

$$= \left|\frac{e^{i\theta}-1}{2}\cdot 1 + i\,\frac{e^{i\theta}+1}{2}\cdot 0\right|^2 \tag{11}$$

$$= \left|\frac{e^{i\theta}-1}{2}\right|^2 \tag{12}$$

Next, we can finalise the Born rule calculation by taking the square of the absolute value of the amplitude (inner product), applying the calculation tips regarding complex numbers and Euler identity to arrive at the detection probabilities in detector 1. This can be done by using either a) the conjugate rule or b) Euler's formula:
a) Solving using conjugate rule:

$$= \frac{1}{4}(e^{i\theta}-1)(e^{-i\theta}-1) = \frac{1}{4}(1 - e^{i\theta} - e^{-i\theta} + 1) \tag{13}$$

Applying Euler's identity we get:

$$= \frac{1}{2}(1 - \cos(\theta)) \tag{14}$$

b) Solving using Euler's formula:

$$= \frac{1}{4}((\cos(\theta) - 1)^2 + sin^2(\theta)) \quad (15)$$

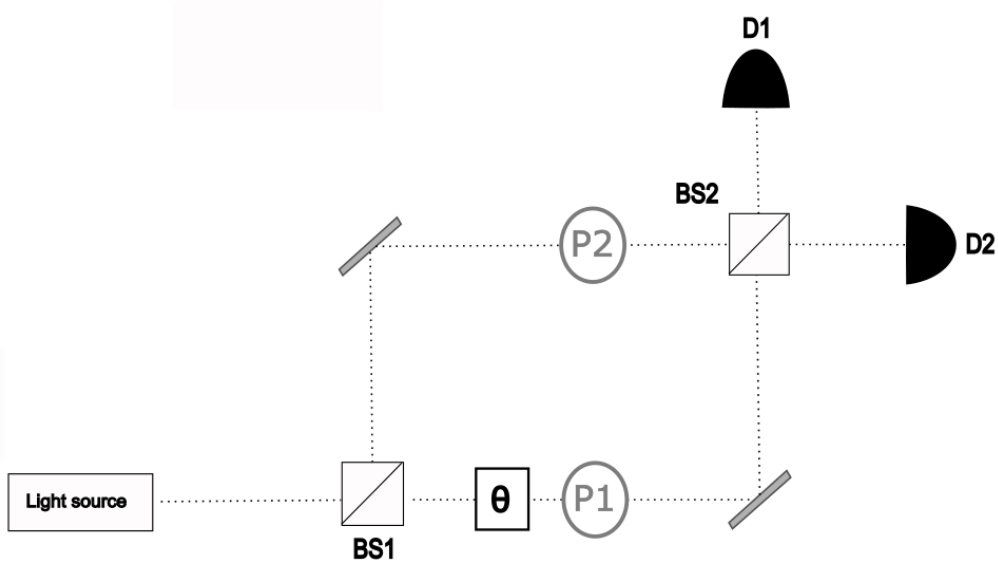
$$= \frac{1}{4}(cos^2(\theta) + sin^2(\theta) + 1 - 2\cos(\theta)) \quad (16)$$

$$= \frac{1}{2}(1 - \cos(\theta)) \quad (17)$$

The same can be applied to detector 2, resulting in a probability of $P_{D2} = \frac{1}{2}(1 + \cos(\theta))$.

Which-way experiment using linear polarisation

Figure 6. Schematic illustration of the Which-way experiment.



We continue the mapping from the previous Eq. 3 by adding the linear polarisers (P1=$|V\rangle$ and P2=$|H\rangle$) to both arms according to rule R6 and R7:

$$\xrightarrow{P1\&P2} \frac{1}{\sqrt{2}}(e^{i\theta}|V\rangle|T\rangle + i|H\rangle|R\rangle) \quad (18)$$

After a second beamsplitter (BS2), applying rules R1, R2, and R3 again, we arrive at the final transformation of the quantum state:

$$\xrightarrow{BS2} \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}e^{i\theta}|V\rangle(|TT\rangle + i|TR\rangle) + \frac{i}{\sqrt{2}}|H\rangle(|RT\rangle + i|RR\rangle)) \quad (19)$$

Identifying spatial modes related to the two detectors according to rule R5, we get the final formulation of the quantum state:

$$|\psi\rangle = \frac{1}{2}\left(e^{i\theta}|V\rangle - |H\rangle\right)|D_1\rangle + \frac{i}{2}\left(e^{i\theta}|V\rangle + |H\rangle\right)|D_2\rangle \quad (20)$$

Now we can implement the Born rule (rule B1) and calculate the probabilities of detection in each detector.

$$P_{V(D1)} = |\langle V|\langle D_1|\psi\rangle|^2 \quad (21)$$

$$= \left|\langle V|\langle D_1|\frac{1}{2}\Big(\big(e^{i\theta}|V\rangle - |H\rangle\big)|D_1\rangle + i\big(e^{i\theta}|V\rangle + |H\rangle\big)|D_2\rangle\Big)\right|^2 \quad (22)$$

Implementing first the orthogonality (rule B2) for detector states.

$$= \left|\langle V|\frac{1}{2}\Big(\big(e^{i\theta}|V\rangle - |H\rangle\big)\langle D_1|D_1\rangle + i\big(e^{i\theta}|V\rangle + |H\rangle\big)\langle D_1|D_2\rangle\Big)\right|^2 \quad (23)$$

$$= \left|\langle V|\frac{1}{2}(e^{i\theta}|V\rangle - |H\rangle)\right|^2 \quad (24)$$

Implementing the orthogonality (rule B2) for polarisation states:

$$= \left|\frac{1}{2}\left(e^{i\theta}\langle V|V\rangle - \langle V|H\rangle\right)\right|^2 \quad (25)$$

$$= \left|\frac{e^{i\theta}}{2}\right|^2 \quad (26)$$

After mathematical calculation we arrive at:

$$P_{V(D1)} = \frac{1}{4} \quad (27)$$

In a similar fashion, we can calculate detection probabilities in the horizontal polarisation state or in the detector 2 state.

Which-way experiment using circular polarisation

We start similarly as in the linear case but now we replace the linear basis with the new circular polarisation basis of $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$. We can modify this to:

$$|\mathrm{H}\rangle = \frac{1}{\sqrt{2}}(|\mathrm{R}\rangle + |\mathrm{L}\rangle) \text{ and } |\mathrm{V}\rangle = \frac{i}{\sqrt{2}}(|\mathrm{L}\rangle - |\mathrm{R}\rangle) \quad (28)$$

Now we can replace the polarisation state from Eq. 18 with a new circular basis:

$$\xrightarrow{qwp} \frac{1}{\sqrt{2}}\left(e^{i\theta}\frac{1}{\sqrt{2}}(|\mathrm{R}\rangle + |\mathrm{L}\rangle)|\mathbf{T}\rangle + \frac{i}{\sqrt{2}}(|\mathrm{L}\rangle - |\mathrm{R}\rangle)|\mathbf{R}\rangle\right) \quad (29)$$

Applying rules R1, R2, and R3 and identifying the spatial modes related to detector states, rule R5:

$$\xrightarrow{BS2} \frac{1}{2}\left(e^{i\theta}\frac{1}{\sqrt{2}}(|\mathrm{R}\rangle + |\mathrm{L}\rangle)(|\mathrm{D}_1\rangle + i|\mathrm{D}_2\rangle) + \frac{i}{\sqrt{2}}(|\mathrm{L}\rangle - |\mathrm{R}\rangle)(i|\mathrm{D}_2\rangle - |\mathrm{D}_1\rangle)\right) \quad (30)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(e^{i\theta}|\mathrm{R}\rangle|\mathrm{D}_1\rangle + ie^{i\theta}|\mathrm{R}\rangle|\mathrm{D}_2\rangle + e^{i\theta}|\mathrm{L}\rangle|\mathrm{D}_1\rangle + ie^{i\theta}|\mathrm{L}\rangle|\mathrm{D}_2\rangle - |L\rangle|\mathrm{D}_2\rangle - i|L\rangle|\mathrm{D}_1\rangle + |R\rangle|\mathrm{D}_2\rangle + i|R\rangle|\mathrm{D}_1\rangle\right) \quad (31)$$

$$\rightarrow \frac{1}{2\sqrt{2}}\left(((e^{i\theta} + i)|R\rangle + (e^{i\theta} - i)|L\rangle)|\mathrm{D}_1\rangle + ((ie^{i\theta} + 1)|R\rangle + (ie^{i\theta} - 1)|L\rangle)|\mathrm{D}_2\rangle\right) \quad (32)$$

Now we have the quantum state of the which-way experiment using circular polarisation. Next, we can implement the Born rule (rule B1) and calculate the probabilities of detection in each detector:

$$P_{L(D1)} = |\langle L|\langle D_1|\psi\rangle|^2 \quad (33)$$

$$= \left|\langle L|\langle D_1|\frac{1}{2\sqrt{2}}(((e^{i\theta} + i)|R\rangle + (e^{i\theta} - i)|L\rangle)|\mathrm{D}_1\rangle + ((ie^{i\theta} + 1)|R\rangle + (ie^{i\theta} - 1)|L\rangle)|\mathrm{D}_2\rangle)\right|^2 \quad (34)$$

Implementing the orthogonality (rule B2) for detector states:

$$= \left|\langle L|\frac{1}{2\sqrt{2}}(((e^{i\theta}+i)|R\rangle+(e^{i\theta}-i)|L\rangle)\langle D_1|D_1\rangle+((ie^{i\theta}+1)|R\rangle+(ie^{i\theta}-1)|L\rangle)\langle D_1|D_2\rangle)\right|^2 \quad (35)$$

$$= \left|\langle L|\frac{1}{2\sqrt{2}}((e^{i\theta}+i)|R\rangle+(e^{i\theta}-i)|L\rangle)\right|^2 \quad (36)$$

Implementing the orthogonality (rule B2) for polarisation states.

$$= \left|\frac{1}{2\sqrt{2}}(e^{i\theta}-i)\right|^2 \quad (37)$$

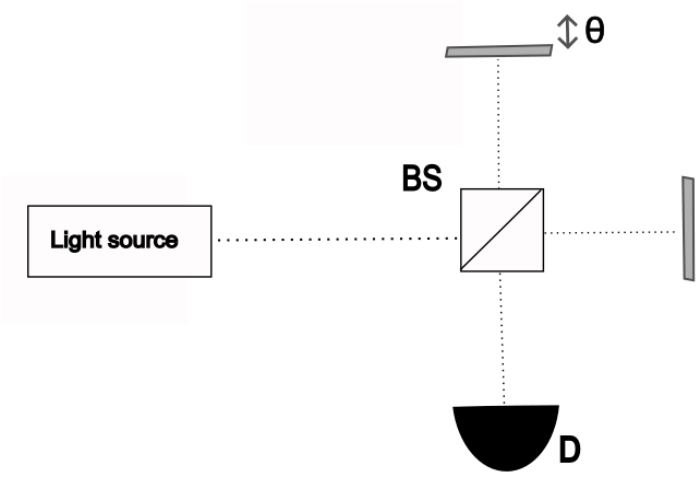
$$= \frac{1}{8}|\cos(\theta)+i\sin(\theta)-i|^2 = \frac{1-\sin(\theta)}{4} \quad (38)$$

The same can be applied to other detectors, resulting in:

$$P_R(D_1) = P_L(D_2) = \frac{1+\sin(\theta)}{4}; \quad P_R(D_2) = P_L(D_1) = \frac{1-\sin(\theta)}{4} \quad (39)$$

Michelson interferometer

Figure 7. Schematic illustration of the Michelson interferometer.



We continue the mapping from the previous Eq. 2. The shifting mirror produces a phase shift (rule R4):

$$\xrightarrow{\theta} \frac{1}{\sqrt{2}}(|T\rangle + e^{i\theta}i\,|R\rangle) \quad (40)$$

After the second beamsplitter (BS2), applying rules R1, R2, and R3 again to obtain the final transformation of the quantum state that becomes:

$$\xrightarrow{BS2} \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|TT\rangle + i|TR\rangle) + \frac{1}{\sqrt{2}}e^{i\theta}i(|RT\rangle + i|RR\rangle)) \quad (41)$$

Identifying spatial modes related to the detector according to rule R5, we get the final form of the quantum state. In this case, a custom notation of $|X\rangle$ is used to denote unmonitored output channels.

$$=\frac{1}{2}(|X\rangle + i|D\rangle + ie^{i\theta}|D\rangle - e^{i\theta}|X\rangle) \tag{42}$$

$$|\psi\rangle = \frac{1-e^{i\theta}}{2}|X\rangle + \frac{1+e^{i\theta}}{2} i|D\rangle \tag{43}$$

We have the quantum state of the Michelson interferometer from Eq. 43. Now we implement the Born rule and calculate the probabilities of detection in the detector (rule B1):

$$P_D = |\langle D|\psi\rangle|^2 = \left|\langle D|(\frac{1-e^{i\theta}}{2}|X\rangle + \frac{1+e^{i\theta}}{2} i|D\rangle)\right|^2 \tag{44}$$

$$= \left|\frac{1-e^{i\theta}}{2}\langle D|X\rangle + \frac{1+e^{i\theta}}{2} i\langle D|D\rangle\right|^2 \tag{45}$$

For the inner product, we implement the orthogonality of states (rule B2):

$$= \left|\frac{1-e^{i\theta}}{2}\cdot 0 + \frac{1+e^{i\theta}}{2} i\cdot 1\right|^2 \tag{46}$$

$$= \frac{1}{2}(1 + \cos(\theta)) \tag{47}$$

References

Bajracharya, R. R., & Thompson, J. R. (2016). Analytical derivation: An epistemic game for solving mathematically based physics problems. *Physical Review Physics Education Research*, *12*(1), 010124. https://doi.org/10.1103/PhysRevPhysEducRes.12.010124

Bielaczyc, K., & Kapur, M. (2010). Playing Epistemic Games in Science and Mathematics Classrooms. *Educational Technology*, *50*.

Bitzenbauer, P. (2021). Practitioners' views on new teaching material for introducing quantum optics in secondary schools. *Physics Education*, *56*(5), 055008. https://doi.org/10.1088/1361-6552/ac0809

Bitzenbauer, P., & Meyn, J.-P. (2021). Fostering students' conceptions about the quantum world – results of an interview study. *Progress in Science Education (PriSE)*, *4*(2), Article 2. https://doi.org/10.25321/prise.2021.1079

Bouchée, T., de Putter - Smits, L., Thurlings, M., & Pepin, B. (2022). Towards a better understanding of conceptual difficulties in introductory quantum physics courses. *Studies in Science Education*, *58*(2), 183–202. https://doi.org/10.1080/03057267.2021.1963579

Bungum, Henriksen, E., Angell, C., Tellefsen, C., & Bøe. (2015). ReleQuant – Improving teaching and learning in quantum physics through educational design research. *Nordina: Nordic Studies in Science Education*, *11*, 153–168. https://doi.org/10.5617/nordina.2043

Chen, Y., Irving, P. W., & Sayre, E. C. (2013). Epistemic game for answer making in learning about hydrostatics. *Physical Review Special Topics - Physics Education Research*, *9*(1), 010108. https://doi.org/10.1103/PhysRevSTPER.9.010108

Collins, A., & Ferguson, W. (1993). Epistemic forms and Epistemic Games: Structures and Strategies to Guide Inquiry. *Educational Psychologist*, *28*(1), 25–42. https://doi.org/10.1207/s15326985ep2801_3

Gerry, C. C., & Knight, P. L. (2004). *Introductory Quantum Optics*. Cambridge University Press.

Gire, E., & Price, E. (2015). Structural features of algebraic quantum notations. *Physical Review Special Topics - Physics Education Research*, *11*(2), 020109. https://doi.org/10.1103/PhysRevSTPER.11.020109

Grangier, P., Roger, G., & Aspect, A. (1986). Experimental Evidence for a Photon Anticorrelation Effect on a Beam Splitter: A New Light on Single-Photon Interferences. *Europhysics Letters*, *1*(4), 173. https://doi.org/10.1209/0295-5075/1/4/004

Hennig, F., Tóth, K., Förster, M., & Bitzenbauer, P. (2024). A new teaching-learning sequence to promote secondary school students' learning of quantum physics using Dirac notation. *Physics Education*, *59*(4), 045007. https://doi.org/10.1088/1361-6552/ad353d

Hennig, F., Tóth, K., Veith, J., & Bitzenbauer, P. (2024a). Introducing quantum physics concepts and Dirac notation at the secondary school level: Insights into student reasoning from an acceptance survey. *Physical Review Physics Education Research*, *20*(2), 020147. https://doi.org/10.1103/PhysRevPhysEducRes.20.020147

Hennig, F., Tóth, K., Veith, J., & Bitzenbauer, P. (2024b). Mathematical sense making of quantum phenomena using Dirac notation: Its effect on secondary school students' functional thinking about photons. *EPJ Quantum Technology*, *11*(1), Article 1. https://doi.org/10.1140/epjqt/s40507-024-00274-6

Hu, D., Chen, K., Leak, A. E., Young, N. T., Santangelo, B., Zwickl, B. M., & Martin, K. N. (2019). Characterizing mathematical problem solving in physics-related workplaces using epistemic games. *Physical Review Physics Education Research*, *15*(2), 020131. https://doi.org/10.1103/PhysRevPhysEducRes.15.020131

Koponen, I., Pajula, A., & Nousiainen, M. (2025). Finnish pre-service physics teachers' opinions about core concepts of quantum optics: Comparison to European multi-stakeholder perspective. *LUMAT-B: International Journal on Math, Science and Technology Education*, *10*(1), 1–1. https://doi.org/https://urn.fi/urn:nbn:fi:hulib:editori:lumatb.v10i1.26000

Lautesse, P., Ferlin, F., Kubik, B., & Tuaillon, J. (2026). Dirac notation: Difficulties of the third-year undergraduate physics students. *European Journal of Physics*, *47*(4), 045706. https://doi.org/10.1088/1361-6404/ae934b

Levy, J., & Singh, C. (2025). Teaching quantum formalism and postulates to first-year undergraduates. *American Journal of Physics*, *93*(1), 46–51. https://doi.org/10.1119/5.0209945

Maries, A., Sayer, R., & Singh, C. (2017). Effectiveness of interactive tutorials in promoting ``which-path'' information reasoning in advanced quantum mechanics. *Physical Review Physics Education Research*, *13*(2), 020115. https://doi.org/10.1103/PhysRevPhysEducRes.13.020115

Marshman, E., & Singh, C. (2016). Interactive tutorial to improve student understanding of single photon experiments involving a Mach–Zehnder interferometer. *European Journal of Physics*, *37*(2), 024001. https://doi.org/10.1088/0143-0807/37/2/024001

Marshman, E., & Singh, C. (2018). Investigating and improving student understanding of quantum mechanical observables and their corresponding operators in Dirac notation. *European Journal of Physics*, *39*(1), 015707. https://doi.org/10.1088/1361-6404/aa8e73

Merzel, A., Bitzenbauer, P., Krijtenburg-Lewerissa, K., Stadermann, K., Andreotti, E., Anttila, D., Bondani, M., Chiofalo, M. L. (Marilù), Faletič, S., Frans, R., Goorney, S., Greinert, F., Jurčić, L., Koupilová, Z., Malgieri, M., Müller, R., Onorato, P., Pospiech, G., Ubben, M., … Pol, H. (2024). The core of secondary level quantum education: A multi-stakeholder perspective. *EPJ Quantum Technology*, *11*(1), 27. https://doi.org/10.1140/epjqt/s40507-024-00237-x

Merzel, A., Weissman, E., Katz, N., & Galili, I. (2024). Mathematical structures in quantum physics education for high school students: Unveiling the power of Dirac notation for conceptual and problem-solving proficiency. *Physical Review Physics Education Research*, *20*. https://doi.org/10.1103/PhysRevPhysEducRes.20.020134

Myhrehagen, H. V., & Bungum, B. (2016). 'From the cat's point of view': Upper secondary physics students' reflections on Schrödinger's thought experiment. *Physics Education*, *51*(5), 055009. https://doi.org/10.1088/0031-9120/51/5/055009

Pereira, A., & Solbes, J. (2022). The Dynamics of Perspective in Quantum Physics. *Science & Education*, *31*(2), 427–450. https://doi.org/10.1007/s11191-021-00252-9

Pospiech, G., Merzel, A., Zuccarini, G., Weissman, E., Katz, N., Galili, I., Santi, L., & Michelini, M. (2021). The Role of Mathematics in Teaching Quantum Physics at High School. In B. Jarosievitz & C. Sükösd (Eds), *Teaching-Learning Contemporary Physics: From Research to Practice* (pp. 47–70). Springer International Publishing. https://doi.org/10.1007/978-3-030-78720-2_4

Riihiluoma, W. D., Topdemir, Z., & Thompson, J. R. (2025). Symbolic forms analysis of expressions for probability in Dirac and wave function notations for spins-first students. *Physical Review Physics Education Research*, *21*(1), 010105. https://doi.org/10.1103/PhysRevPhysEducRes.21.010105

Scarani, V., Lynn, C., Yang, L. S., & Yan, H. J. (2010). *Six Quantum Pieces: A First Course in Quantum Physics* (p. 140). https://doi.org/10.1142/7965

Scarani, V., & Suarez, A. (1998). Introducing quantum mechanics: One-particle interferences. *American Journal of Physics - AMER J PHYS*, *66*, 718–721. https://doi.org/10.1119/1.18938

Singh, C., & Marshman, E. (2013). *Investigating Student Difficulties with Dirac Notation*. 345–348. https://www.per-central.org/items/detail.cfm?ID=13149

Tuminaro, J., & Redish, E. (2005). *Students ' Use of Mathematics in the Context of Physics Problem Solving: A Cognitive Model*. https://www.semanticscholar.org/paper/Students-%E2%80%99-Use-of-Mathematics-in-the-Context-of-%3A-A-Tuminaro-Redish/da5eccb838885d04b9df780ede123f3be48bb76e

Tuminaro, J., & Redish, E. F. (2007). Elements of a cognitive model of physics problem solving: Epistemic games. *Physical Review Special Topics - Physics Education Research*, *3*(2), 020101. https://doi.org/10.1103/PhysRevSTPER.3.020101

Zuccarini, G., Sutrini, C., Bondani, M., Macchiavello, C., & Malgieri, M. (2024). Teaching quantum information science to secondary school students with photon polarization and which-path encoding. *EPJ Quantum Technology*, *11*(1), 74. https://doi.org/10.1140/epjqt/s40507-024-00287-1