



# LUMAT

**Special Issue:  
Mathematical Thinking  
and Understanding in  
Learning of Mathematics**

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## Mathematical thinking and understanding in learning of mathematics

We both editors have wondered and studied “*What is mathematical thinking?*” more than thirty years. At least the question could be recognized behind most of our research projects concerning studies in mathematics education from little school children to university students. The concept “mathematical thinking” can be found in several studies of mathematics education, in national curricula or in media during the decades all over the world. We searched words “mathematical thinking” from the database of international scientific articles, and we found 456 707 mentions at first time. These are the main reasons why we have chosen “mathematical thinking” as the central concept of the special issue. The other interesting question from our point of view is how a student can express his/her mathematical thinking? By answering this question, we have made simple model for the teacher education purposes, and we call it “*languaging*” (of mathematical thinking). In the following, we lead to the above concepts and prepare the presentation of articles in this journal.

Sternberg (1996) has studied different approaches to the concept of mathematical thinking. He found at least five different points of view to describe the concept. They are anthropological, information process, mathematical, pedagogical, and psychometric approach. For example, in the anthropological approach the central starting point is the surrounding culture (e.g. ethnomathematics d’Ambrosio, 1985), in the information process different types of knowledge in mathematics (e.g. Joutsenlahti, 2009) or in psychometric the abilities in doing mathematics (e.g. Krutetskii, 1976). We can interpret that the pedagogical approach in the school context takes account on beliefs and problem solving (e.g. Pehkonen, 1998, 2007 and Hannula, 2004) in thinking processes. We have used the information process approach in describing the concept of mathematical thinking, and we described knowledge (meaningful information) as conceptual and procedural (Hiebert & Lefevre, 1986). Student’s metacognition guides his/her thinking. Oikkonen and Hannula have taken the viewpoint to mathematical thinking David Tall’s framework of the three worlds of mathematics in their article “*The three worlds and two sides of mathematics and a visual construction for a continuous nowhere differentiable function*”. In their theoretical article, they further elaborate Tall’s framework and demonstrate this framework by discussion on the definition of continuity. Kayan Fadlelmula’s article “*A PRISMA Systematic Review on Enablers and Obstacles in Teaching and Learning of Mathematics*”



is systematic review on the current issues positively and negatively affecting teaching and learning in mathematics and the data was gathered from the studies published in the LUMAT -journal. Metsämuuronen's and Ukkola's article "*Rudimentary stages of the mathematical thinking and proficiency - Mathematical skills of low-performing pupils at the beginning of the first grade*" based on the national-level dataset (n = 7770) at grade 1 of primary school in Finland and the focus is on those pupils whose preconditions are so low that they are below the first measurable level of proficiency in the common framework with reference to mathematics.

When we were young teachers, we often thought ***How we can express mathematical thinking?*** and especially how we could encourage students to do it by many ways? Traditionally, in mathematics classes, students work quietly with their own textbook and asked for help only from the teacher. We didn't often know what kind of thoughts our students had about the solutions processes of mathematics problems. Nevertheless, we finally understood that if we get a student to speak about mathematics – we get him/her to think mathematics and we can hear his/her mathematical thinking! Also, we can see it if the student does it by writing or/and drawing (see e.g. Morgan, 2001). We call this process languaging of mathematical thinking, which is based on a model of four "languages". They are mathematical symbolic language, natural language, pictorial language, and tactile action language (Joutsenlahti & Kulju, 2017; Joutsenlahti & Perkkilä, 2019). The most effective benefit of languaging for the student is that when the student expresses mathematical thoughts by his/her own words then he/she structures his/her thinking and by that way understands mathematical concepts and procedures better. It is for the teacher easy to evaluate student's thinking and give help if needed. When a student expresses his/her mathematical thinking (s)he can use different multimodal approaches (e.g. the four "languages"). Theoretically, the multimodal languaging model is related to multiliteracy (Kalanzis & Cope, 2012). When a student makes meanings for the mathematical text the languages can be seen as a multi-semiotic approach, where the different languages make it possible to construct many kinds of meanings for concepts in versatile contexts (Joutsenlahti & Perkkilä, 2019). Björklund's, Ekdahl's, Kullberg's and Reis's article "*Preschoolers' ways of experiencing numbers*" directs attention to 5–6-year-olds' learning of arithmetic skills through a thorough analysis of changes in the children's ways of encountering and experiencing numbers. The aim of Kaitera's and Harmoinen's study was to map whether a teaching approach, which focuses on teaching general heuristics for mathematical problem-solving by providing visual tools called



Problem-solving Keys, would improve students' performance in tasks and skills in justifying their reasoning in their article "*Developing mathematical problem-solving skills in primary school by using visual representations on heuristics*". Francisco's article "*Supporting Argumentation in Mathematics Classrooms: The Role of Teachers' Mathematical Knowledge*" addresses a documented need for a better understanding of the relationship between mathematical knowledge for teaching and instruction by focusing on how the knowledge influences teachers' support of argumentation. Rinneheimö concentrates on the use of languaging exercises in the engineering mathematics course in Finland in her article "*I solved a derivative – but what does it actually mean? Languaging and conceptual understanding in engineering mathematics.*"

We have thought about the relationship between **conceptual understanding and mathematical thinking**. The development of mathematical thinking is emphasized in the Finnish curricula of pre-school and school education. The main goal of the curricula is to develop a student's mathematical thinking and understanding about mathematics. Hiebert and Lefevre (1986, p. 3–8) have defined conceptual and procedural mathematical knowledge. According to them, procedural knowledge refers to those procedures that are needed to solve mathematical tasks and problems. Conceptual knowledge can be described as the richness of knowledge in relationships between things which 'can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions, so that all pieces of information are linked to some network' (Hiebert & Lefevre, 1986, pp 3–4). Both definitions of procedural and conceptual knowledge share commonalities with Skemp's definitions of similar concepts (Skemp, 1976). Hiebert and Lefevre's (1986) description of procedural knowledge resembles the definition of instrumental knowledge by Skemp (1976), which can be seen as the application of finished formulas and models to certain kinds of tasks. In the definitions of conceptual knowledge, both Hiebert and Lefevre (1986) and Skemp emphasize understanding about the connections made by mathematical concepts. When these connections between concepts are built purposefully in teaching, students gradually develop an understanding about the network of mathematical concepts. Thus, a student does not use loose mathematical concepts; (s)he understands the whole system of them. The interactivity of the learning environment, student's timely support and received feedback and the process of becoming accepted as oneself contributes to the construction of a sustainable first-hand

mathematical knowledge and skills base, i.e. the building of mathematical competence. It is important that the learning community (teachers and students) have a feeling let's do it together, talk and model mathematical solutions through the means of languaging.

Early mathematical skills build a foundation for the individual's comprehending learning of school mathematics skills and mathematical knowledge. The level of development and mathematical knowledge of students' early mathematical skills meet no later than preschool. In order to develop the student's mathematical thinking skills, we need to understand how (s)he learns mathematics. Preschool age and elementary school students are on the concrete level of mathematical thinking, and it is reflected in their actions. Conceptual understanding develops best in a sociocultural context by collaborative working methods where students construct their own mathematical thinking through drawings, using mathematical symbolic language, concrete and verbal actions (e.g. Perkkilä & Joutsenlahti, 2021). This viewpoint is in line with Vygotsky's theory, which emphasizes the sociocultural perspective. Building a math-speaking community where everyone is a teacher and learner is crucial for students building a conceptual network in a particular math area. (e.g. Fuson, 2019.) This allows all students from preschool to university to build their own mathematical thinking from their own mathematical skill level. By supporting the construction of student's mathematical thinking and conceptual understanding, we support sustainable development from the perspective of learning mathematics (e.g. Joutsenlahti & Perkkilä, 2019; Perkkilä & Joutsenlahti, 2021).

Algebraic thinking is an important part of mathematical thinking. Both Sanna Wettergren and Inger Eriksson and Natalia Tabachnikova have studied how to promote young students' algebraic thinking in their articles. Wettergren explored how teaching aiming to promote young students' algebraic thinking can be designed in her article "*Identifying and promoting young students' early algebraic thinking*". Eriksson and Tabachnikova have sought answers for the development of algebraic thinking with an example based on a case study that describes how young students can theoretically study and reflect some aspects of the equations in their article "*IE "Learning models": utilising young students' algebraic understanding of equations*". Kambara's and Tossavainen's articles focus on examining conceptual understanding in students studying to be a teacher. Kambara's article "*Understanding of "proportion" and mathematical identity: A study of Japanese elementary school teachers*" explores and clarifies the level of conceptual understanding of "proportions" among

Japanese students who hope to become elementary school teachers in the future. Tossavainen's article "*Student Teachers' Common Content Knowledge for Solving Routine Fraction Tasks*" focuses on the knowledge base that Swedish elementary student teachers demonstrate in their solutions for six routine fraction tasks.

We think that we have got very good sample of scientific articles to our Special Issue. Thank you for all the writers, you have done excellent work!

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Guest Editors

Jorma Joutsenlahti and Päivi Perkkilä

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# The three worlds and two sides of mathematics and a visual construction for a continuous nowhere differentiable function

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A rigorous and axiomatic-deductive approach is emphasized in teaching mathematics at university-level. Therefore, the secondary-tertiary transition includes a major change in mathematical thinking. One viewpoint to examine such elements of mathematical thinking is David Tall's framework of the three worlds of mathematics. Tall's framework describes the aspects and the development of mathematical thinking from early childhood to university-level mathematics. In this theoretical article, we further elaborate Tall's framework. First, we present a division between the subjective-social and objective sides of mathematics. Then, we combine Tall's distinction to ours and present a framework of six dimensions of mathematics. We demonstrate this framework by discussion on the definition of continuity and by presenting a visual construction of a nowhere differentiable function and analyzing the way in which this construction is communicated visually. In this connection, we discuss the importance to distinguish the subjective-social from the objective side of mathematics. We argue that the framework presented in this paper can be useful in developing mathematics teaching at all levels and can be applied in educational research to analyze mathematical communication in authentic situations.

Keywords: mathematical thinking, advanced mathematics, the three worlds of mathematics, secondary-tertiary transition, nowhere differentiable functions

## 1 Introduction

Research on mathematical thinking includes several approaches that focus on different aspects of the subject such as the pedagogical, cultural or cognitive (Sternberg, 1996). A recent review of research on mathematical thinking (Goos & Kaya, 2020) divides these different approaches into individual cognitive and constructivist perspective, cultural psychology perspective and discourse perspective. Mathematics education research has a long tradition in exploring the cognitive aspects of mathematical thinking (see e.g., Bingolbali & Monaghan, 2008; Fan & Bokhove, 2014; Tall, 1991). In this theoretical article, we draw upon that tradition and present a novel theoretical framework describing various aspects of mathematical thinking in mathematical discourse. We discuss the framework using examples from

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university-level mathematics, although we also consider the significance of our theoretical elaboration for primary and secondary school mathematics teaching and learning.

Several studies have described special features of the context of university mathematics. One of the main interests of such studies has been the secondary-tertiary transition. The transition includes a change in mathematical content, sociomathematical norms and educational culture (Education Committee of the EMS, 2013), and therefore causes both cognitive and pedagogical shocks to beginning undergraduates (Clark & Lovric, 2009). The secondary-tertiary transition has been found problematic for decades and consequently many beginning undergraduate become dropouts (Di Martino & Gregorio, 2019). Regarding cognitive aspects of the transition, a rigorous and axiomatic-deductive approach is emphasized at university, meaning that the transition includes a major change in mathematical thinking (Tall, 2008). For this reason, some universities have, for example, developed special bridging courses to ease the transition to advanced mathematical thinking.

Mathematics, however, is not only an art of axiomatic-deductive reasoning or manipulating symbols, neither in school mathematics nor in university-level mathematics. One influential framework to describe the variety of mathematical thinking is the framework of the three worlds of mathematics (Tall, 2013). Tall (2013) divides mathematical thinking into embodied world (pictures, gestures etc.), symbolic world (calculations, symbolic rules etc.) and formal world (axioms, proofs etc.). The interplay between these different worlds of mathematics has been found useful for developing undergraduate level mathematics teaching (Oikkonen, 2009), as well as teacher education (Hannula, 2018). That is to say, such interplay is important in terms of secondary-tertiary transition, as well as in terms of development of pre-service teachers' mathematical knowledge for teaching (see e.g., Dreher & Kuntze, 2015).

In this article, we elaborate the framework of three worlds of mathematics further. First, we shall discuss Tall's framework and combine it with a distinction of two sides of mathematics, that is, the subjective-social and objective sides of mathematics. Together these ways of looking at mathematics will lead to a division of  $6 = 2 \times 3$  dimensions of mathematics. After discussing mathematical thinking in general, we present a discussion clarifying the definition of continuity and a construction for a continuous nowhere differentiable function. Such functions are related to advanced undergraduate level mathematics courses. The construction, and the way in which we present it, is a novel one and does not appear for instance in Thim's extensive review

(2003) of continuous nowhere differentiable functions. David Tall has been using a construction which he calls the Blancmange function in many of his writings (see e.g., Tall & Di Giacomo, 2000; Tall, 1982). We use our construction as an example to demonstrate six different dimensions of mathematics and their interplay.

The motivation of our theoretical considerations is connected to our experiences as a research mathematician and university lecturer (the first author) and as a mathematics teacher and teacher educator (the second author). In our work, we have found Tall's framework extremely fruitful in developing teaching and conducting educational research. However, we have come to the conclusion that Tall's distinction does not capture all aspects of mathematical discourse in authentic situations. Therefore, we find a theoretical elaboration of Tall's framework useful in developing teaching and in educational research. In this article, we aim to

1. introduce a novel theoretical framework of the six dimensions of mathematics
2. demonstrate the framework in the cases of the definition of continuity and a construction of a continuous nowhere differentiable function
3. discuss the possibilities of the framework for educational research and development of mathematics teaching and learning.

## 2 Theoretical framework

We examine the broad concept of mathematical thinking from a cognitive viewpoint. In the following subsections, we first present a summary of frameworks describing cognitive aspects of mathematical thinking. Second, we discuss in more detail the framework of the three worlds of mathematics (Tall, 2013). Third, we present our distinction between subjective-social and objective sides of mathematics. Finally, we elaborate Tall's framework further by combining the three worlds of mathematics with the distinction of two sides of mathematics.

### 2.1 Cognitive frameworks of mathematical thinking

Since the very beginning of the discipline, mathematics education researchers have presented several dichotomies and classifications of mathematical thinking and knowledge. Skemp (1976), for instance, divides mathematical understanding into *instrumental* understanding and *relational* understanding. Roughly speaking, instrumental understanding refers to *how* to carry out mathematical operations whereas relational understanding refers to *why* mathematical operations work.

Similarly, Hiebert (1986) divides between *conceptual* and *procedural* knowledge. Several similar dichotomies have been used in educational theories, and Haapasalo (2003), for instance, lists 20 such dichotomies presented in literature. Instead of discussing all these dichotomies, we summarize some of the most influential frameworks of mathematical thinking underlying present mathematics education research.

One of the most established frameworks in mathematics education research is the distinction between *concept image* and *concept definition* (Tall & Vinner, 1981). (It seems that the origin of David Tall's three worlds of mathematics lies there.) Concept image refers roughly to one's understanding of a mathematical concept and concept definition to the official definition of the concept. Tall and Vinner (1981) define concept image as the total cognitive structure that is associated with a mathematical concept. Thus, concept image may include mental pictures, symbolic processes and axioms etc.

Development of students' concept images have been widely studied in literature especially from the viewpoint of processes and concepts. The term *encapsulation*, originating from Piaget, means a change in thinking in which learner starts to think the concept itself instead of the process (Tall, 2013). For instance, Sfard (1991) considers the dualism between the operational and structural sides of mathematics. The encapsulation process, according to Sfard (1991), occurs in three steps: interiorization, condensation and reification. Similarly, Gray and Tall (1991) speak of *procepts* referring to an 'amalgam of process and concept in which process and product is represented by the same symbolism' (Gray & Tall, 1991, p. 73). Additionally, one influential framework explaining the encapsulation process is APOS-theory presented by Ed Dubinsky and colleagues (Asiala et al., 1996; Dubinsky & McDonald, 2002).

Frameworks presented above focus mostly on learning of algebra and calculus. In case of geometry, for instance, van Hiele levels (Burger & Shaughnessy, 1986) give a widely used framework to analyze students' learning. Some researchers have, however, presented more generic frameworks. Already in the 1960's Bruner (1967) divided mathematical representations into enactive, iconic and symbolic. Similarly, Fishbein (1994) classifies intuitive, algorithmic and formal approach to mathematical activity.

Furthermore, Viholainen (2008) separates mathematical reasoning into formal reasoning based on axioms, definitions and proven theorems, and informal reasoning

based on visual or physical interpretations of mathematical concepts. Some researchers, such as Joutsenlahti (2005), have studied the overall picture of mathematical thinking including students' knowledge as well as their beliefs. In his doctoral dissertation, Joutsenlahti (2005) explored mathematical thinking from societal perspective, teacher's perspective and student's perspective.

Influenced by many frameworks presented above, Tall presented his framework of the three worlds of mathematics first in a conference paper (Tall, 2004). In that paper Tall divides mathematical thinking into *embodied*, *symbolic* and *formal* worlds of mathematics. Tall's framework aims to give an overall view to mathematical thinking and its development (Tall, 2013). Adapted from Chin (2013), some established frameworks of mathematical thinking are summarized in Table 1.

**Table 1.** Summary of frameworks adapted from Chin (2013)

Researcher(s)	Key concepts of the framework	Focus
Sfard	operational – structural	encapsulation process
Gray & Tall	procept	procedural and conceptual knowledge
Dubinsky et al.	action - process - object - schema	cognitive development
Van Hiele	perceptions - operations - proofs	levels of knowledge in geometry
Bruner	iconic - enactive - symbolic	representations
Fischbein	intuitive - algorithmic - formal	approaches to mathematics
Viholainen	informal - formal	mathematical argumentation and reasoning
Joutsenlahti	knowledge - beliefs	aspects of mathematical thinking
Tall	embodiment - symbolism - formalism	modes of mathematical thinking

The summary in Table 1 highlights the variety of frameworks describing cognitive aspects of mathematical thinking. In this article, we elaborate Tall's broad framework of the three worlds of mathematics.

## 2.2 The three worlds of mathematics

The idea of three worlds of mathematics is based on humans' capability to

- i) *recognize* regularities, similarities and differences,
- ii) *repeat* actions, and
- iii) *use language* to name concepts (Tall, 2013).



Based on these humans' cognitive-physiological capabilities Tall divides mathematical thinking into conceptual-embodied, proceptual-symbolic, and axiomatic-formal worlds of mathematics. Tall has discussed his worlds in a great number of writings and the concepts have developed somewhat over the years.

In his book Tall (2013) gives an overall view of his work and describes the worlds as follows.

A world of (conceptual) embodiment building on human perceptions and actions developing mental images verbalized in increasingly sophisticated ways to become perfect mental entities in our imagination;

A world of (operational) symbolism developing from embodied human actions into symbolic procedures of calculation and manipulation that may be compressed into procepts to enable flexible operational thinking;

A world of (axiomatic) formalism building formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof. (Tall, 2013, p. 133)

Later, we refer to these worlds simply as embodied, symbolic and formal.

The embodied world includes embodied thinking about mathematical concepts and processes such as pictures and physical objects, whereas the symbolic world includes symbolic thinking such as calculation rules. Formal world, on the other hand, includes rigorous mathematical theory including proofs and axioms. As an example, Tall (2013, p. 25) relates the system of the real numbers to these worlds. The real numbers have embodiment as a number line, symbolism as (infinite) decimals, and formalism as a complete ordered field (Figure 1).

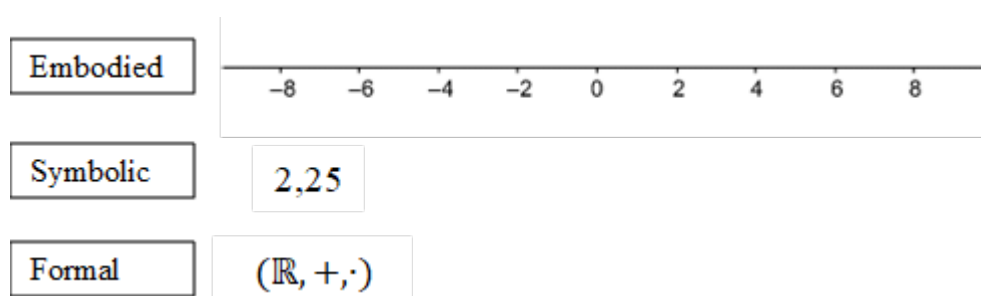


Figure 1. The concept of real number and the three worlds of mathematics

Although all these worlds are apparent in both school and university mathematics, the secondary-tertiary transition includes a change in emphasis from the embodied and symbolic world to the formal world (Tall, 2004; Tall, 2008). Therefore, the interplay between the different worlds is crucial in undergraduate mathematics and

teacher education (Oikkonen, 2009; Hannula, 2018). Although the worlds are hierarchical in regard to cognitive development, all of them are more or less present in mathematical discourse at all levels of education.

Sfard's (1991) framework has some resemblance to the three worlds of Tall, but the ordering makes a big difference. The doctoral thesis of Hähkiöniemi (2006) is interesting in this respect. Tall (2008) remarks that Hähkiöniemi (2006) considered the routes of students towards learning the derivative. Tall say that Hähkiöniemi 'found that the embodied world offers powerful thinking tools for students' who 'consider the derivative as an object at an early stage'. According to Tall this questions Sfrad's suggestion that operational thinking precedes structural.

### 2.3 The subjective-social and objective sides of mathematics

As discussed above, several dichotomies and distinctions of mathematical thinking and activity have been presented in literature. These frameworks focus, for instance, on representations and cognitive development (e.g. Bruner, 1967; Fischbein, 1994). On the other hand, many of these frameworks somehow distinct between formal and informal aspects (e.g. Tall, 2013; Viholainen, 2008) or conceptual and procedural aspects (e.g. Gray & Tall, 1991) of mathematical thinking. Our distinction, presented in this section, is somewhat different to prior distinctions, and can actually be seen as 'orthogonal' to many of them.

Our distinction is based on the observation that there are aspects in mathematics that are objective and others that are subjective or social. To the first belong printed formulas and pictures that one can find in textbooks etc. To the latter belong my mental images that I as the author had in my mind while writing formulas or making pictures appearing in printed material, and your mental images that you as a reader had in your mind while reading the text. We refer to this distinction by speaking about the two sides of mathematics.

In our own work as university and schoolteachers, as well as mathematics and mathematics education researchers, such a division between two sides of mathematics has become important. But the emphasis seems to be somewhat different from those approaches referred to above. For us the division is related to what one does 'here and now' e.g., while working on a mathematical problem or teaching a mathematical concept: does one in the next moment speak about the ideas behind a mathematical concept or does one explicitly work with the formal definition of the concept. Our idea of the two sides was initially outlined several years ago

(Oikkonen, 2004) and it has been an important idea behind first author's development of university mathematics teaching (Oikkonen, 2008, 2009).

Let us consider as an example the continuity of a function  $g$  at a point  $a$ . The idea is simple:  $g(x)$  should be near  $g(a)$  when  $x$  is near  $a$ . This is often visualized by well-known pictures like Figure 2.

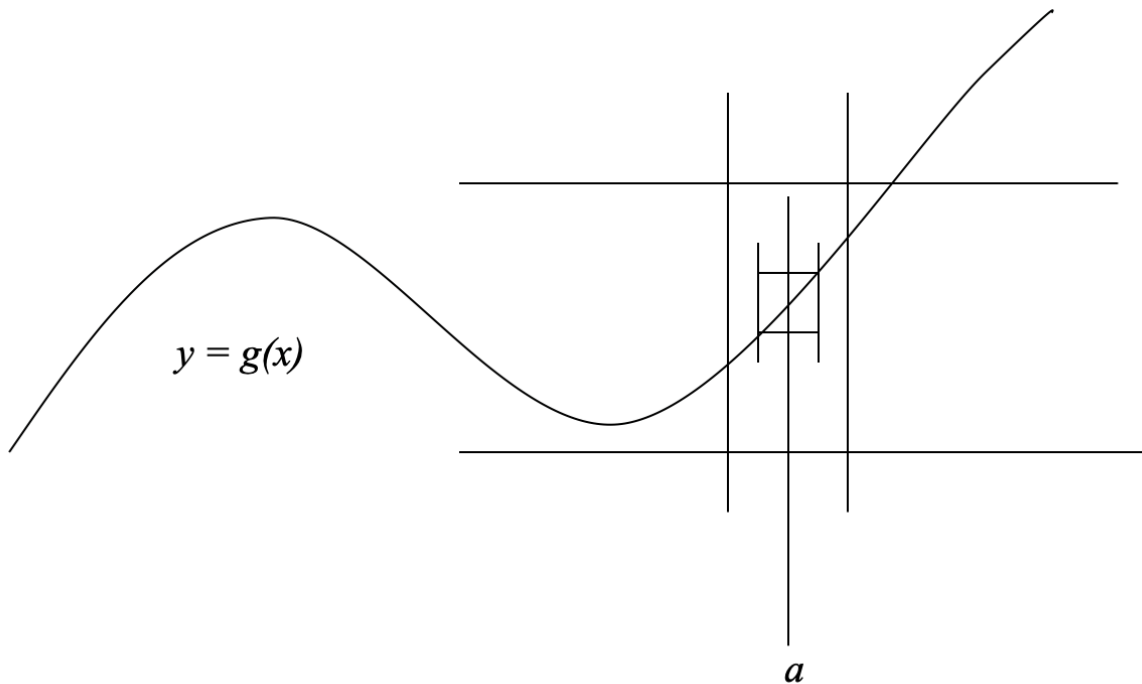


Figure 2. Continuity of  $g$  at  $a$

Pictures like that in Figure 2 can be argued to represent the embodied world of mathematics while the exact epsilon-delta definition represents symbolic world of mathematics. But if we look closer at how these pictures are used in teaching, we see an example of the interplay between the subjective-social and the objective sides of mathematics.

We come now to our first main example. Consider an imaginary discussion between a teacher T and (a) student(s) S. The letters A to E refer to the pictures in Figure 3 (A-E).

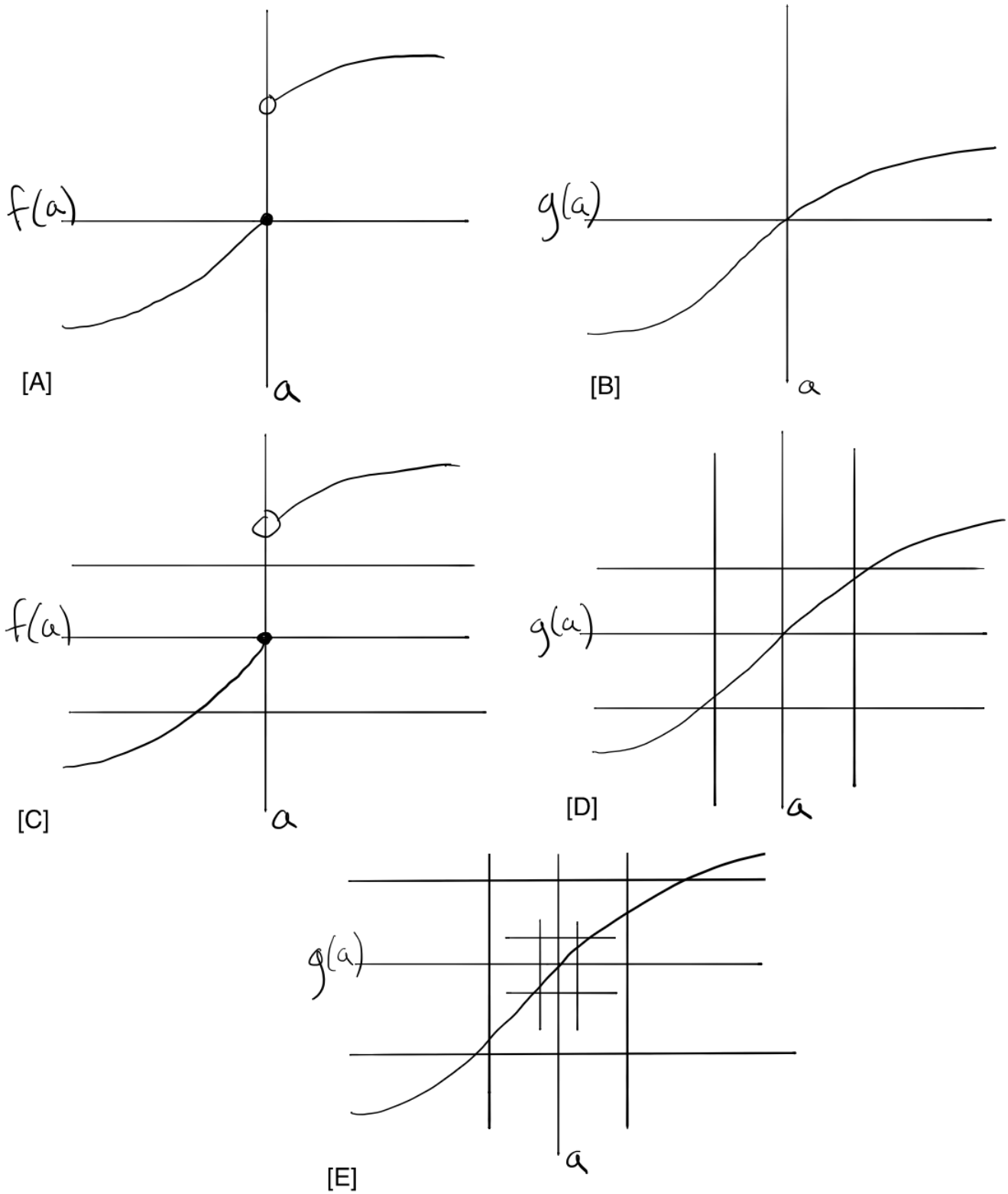


Figure 3. Discussion about continuity

The discussion goes in the following way.

T: Is  $f$  appearing in picture A continuous at  $a$ ?

S: No!

T: Is  $g$  in  $B$  continuous at  $a$ ?

S: Yes!

T: Why  $f$  is not continuous, but  $g$  is continuous?

S:  $g(x)$  goes near  $g(a)$  as  $x$  goes near  $a$  but  $f(x)$  does not go near  $f(a)$  as  $x$  goes near  $a$ .

T: Can you elaborate / be more exact?

S: ???

T: Let us draw horizontal lines shown in  $C$  and  $D$ . What can you see?

S:  $g(x)$  appears between these lines as  $x$  is close to  $a$  but a part of  $f$  stays outside the lines no matter how close to  $a$  we are.

T: Yes! For  $g$  we get in the bigger rectangle shown in  $E$ . (The graph of)  $g$  does not cut / go through the floor or roof of this rectangle. What happens if we draw new horizontal lines as in  $E$  closer to  $y = g(a)$ ?

S: We can draw new vertical lines and get a smaller rectangle so that  $g$  does not cut the floor or the roof. This appears in the smaller rectangle of  $E$ .

T: Good! When the horizontal lines are near enough the line  $y = g(a)$  so that we can be sure of continuity?

S: ???

T: Never. The point in continuity is that no matter however close we draw the lines, there always are the vertical lines making a box such that  $g$  does not cut the floor or roof of the box. Can you say this in other words?

S: Could it work to say that for all horizontal lines...?

T: Yes. And it is enough to speak about vertical and horizontal distances. Actually, this is exactly what the epsilon-delta definition in your textbook says!

Pictures somewhat like [Figure 2](#) appeared in the above discussion but here they had a role as a means of sharing thinking between the teacher and the student(s). Hence these pictures and the whole discussion are examples of the subjective-social side of mathematics. The discussion ends in a reference to the textbook of the students and the formal definition continuity. These are of course examples of the objective side of mathematics.



When combining the subjective-social vs. objective dichotomy to Tall's three worlds, we can say that the main part of the previous discussion lives on the subjective-social side and in Tall's embodied world.

Continuity has also an objective aspect, the well-known epsilon delta definition appearing in textbooks and mentioned at the end of the above discussion. In Tall's terminology, the definition belongs to the symbolic world, perhaps with a flavor of formal world. According to the definition, a function  $g$  is continuous at  $a$ , if (and only if) for every  $\varepsilon > 0$  there is such a  $\delta > 0$  that  $|g(x) - g(a)| < \varepsilon$  for all  $x$  satisfying  $|x - a| < \delta$ .

One of the main purposes of an introductory course in analysis is to teach this kind of definitions and proofs of the main theorems of analysis based on such definitions. But it is not an easy task. This is not helped by the way how we too often begin solutions of examples or proofs of theorems: 'Assume that  $\varepsilon > 0$ . Let  $\delta = \frac{3}{7}\varepsilon$ ...'

The first author's experience in teaching analysis supports the idea that it is helpful to change the viewpoint from which we look at mathematics. This takes place by combining the formal definitions with an active use of teacher's and students' mental images like the one described above. By doing this it is also possible to reveal in teaching the way in which an expert mathematician thinks.

In our experience this kind of an approach helps in making the content of a mathematics course meaningful and understandable to students. Thus, a course in mathematics is not only the polished formal content of the course but also – and to the authors essentially – the thinking and culture that lies behind the text. We believe that this approach explains partially the success shown in Oikkonen (2009). (There are also other pedagogical ideas involved in this paper.)

The first author's path to this kind of an approach results from the striking similarity between two seemingly quite different types of discussion on mathematics in which he has taken part: those taking place in math days in elementary schools and those taking place when experts discuss some problem in research mathematics. The 'here and now' choice between different kinds of action that was mentioned above seems to be characteristic to such discourses.

So, we have two sides of mathematics. But which of them is the correct one? Let us go back to continuity: which side is the correct one, the human (mental) images or the formal epsilon-delta definition? Our own answer is that neither of them is the correct one. The concept of continuity depends on both of its sides, and it is to us really a kind of interplay between these two sides.

## 2.4 There are six = $2 \times 3$ dimensions of mathematics

Above we discussed the ideas of the tree worlds of David Tall and the two sides of mathematics. In this section, we elaborate on how these ideas can be combined into a new way of looking at mathematics and mathematical thinking. We argue that this leads to new insight in mathematical thinking and communication. It may also help in better understanding of Tall's three worlds.

It seems to us that our distinction between subjective-social and objective and Tall's division between embodied, symbolic and formal look at similar features in mathematical thinking from two different standpoints. Moreover, the resulting  $2 \times 3 = 6$  dimensions of mathematics help us to see some aspects more easily. Indeed, we shall consider some examples that show how each of Tall's three worlds seems to divide into two sides.

The case of the embodied world seems especially natural. Our own mental images of mathematical objects or situations are subjective embodiment. It becomes social when a group of people shares such images while working on a problem. Various objects like number sticks etc. made for teaching mathematics are examples of objective embodiment.

A number line was mentioned above as an embodied version of the system of the real numbers. It can belong to either side depending on what we actually mean. The idea of a line of numbers belongs to the subjective-social side whereas an actual line drawn on a blackboard belongs to the objective side.

But is the real line itself an objective 'mathematical object' belonging to the objective side of mathematics? What do we think about it and its existence? In a sense this is not an important question here. On the subjective-social side most mathematicians seem to behave as if the real line would actually 'be there'. But to us, it seems that we cannot distinguish those mathematicians who really believe that the real line "is there in a Platonic universe" from those who only behave as if it existed. The theorems concerning the reals are proved using the axioms of the reals in the objective side of Tall's formal world and they make no direct reference to the truth or meaning of the actual statement that the 'reals exist'. In this sense formalism and platonism are not very far from each other.

Moreover, it is not clear how to reply from a set-theoretic point of view to the question what the real line really is. Namely, there are different constructions (Dedekind-cuts of the rationals, certain equivalence classes of Cauchy sequences of the rationals etc.) leading to different sets.

The case of the symbolic world is more interesting. Rules for manipulating symbols and correct application of such rules belong to the objective side. These include long divisions in elementary school or solving equations or doing differentiation of expressions for functions in upper secondary school. Students' own minitheories and systematic errors seem to belong to the subjective-social side of the the symbolic world.

Perhaps also various routines applied in what is called street mathematics (see e.g. Resnick, 1995) in basic calculations can be seen also as examples of the subjective-social side of the symbolic world.

Written university level mathematics with its axioms, definitions and theorems is an example of objective side of the formal world of mathematics. Higher level strategic discussion on research mathematics belongs to the subjective-social side or the formal world. An example of this represents the comment 'she mixed ideas from physics to analysis to solve the problem'.

The step from the subjective-social side or the formal world to the subjective-social side of embodied mathematics with its mental images and gestures is very short. A nice example of this is in the Introduction of W. Hodges' book (1985) where he tells about a difficulty with his own doctoral thesis. His supervisor C. C. Chang made an up and down movement with his hand and said: 'This should help.' (see Figure 4) According to Hodges, it helped.



Figure 4. Supervisor's advice

Chang's gesture indicated a certain model theoretic back-and-forth construction and obviously Hodges understood Chang's suggestion. (Such constructions are the main theme of Hodges' book.)

Before leaving this section, we shall have closer look at the concept of continuity of a function discussed above in connection to our two sides of mathematics. There we considered the appearing in Figure 2.

The notion of continuity and the function studied is clearly embodied in such a drawing. (Of course, it is possible that there is no specific function that is considered

and that the whole discussion concerns the concept of continuity.) This drawing is clearly objective in the sense that everybody can observe it. So, the drawing belongs to the objective side and to the embodied world of mathematics.

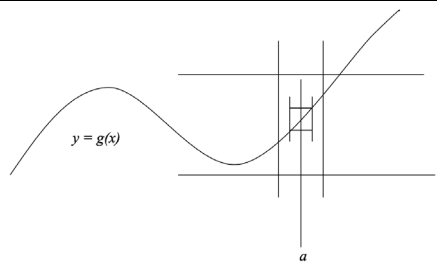
But these drawings are used either by oneself to think about continuity or by a group of people to discuss continuity. Such actions belong to the subjective-social side of the embodied world of mathematics.

When one works with examples of assertions concerning continuity, one usually has to manipulate mathematical formulas. As long as one thinks or discusses how to proceed, one acts in the subjective-social side of the symbolic world of mathematics. When these formulas are actually written they become observable and thus objective and so one acts in the objective side of the symbolic world of mathematics.

But usually, the real interest lies in understanding, teaching or using the ‘epsilon-delta’ -definition of continuity, and so the subjective-social or objective side of Tall’s formal world is involved.

As a conclusion, while discussing the continuity of a function, all six dimensions of mathematics may be involved (Table 2).

**Table 2.** The six dimensions of mathematics in the case of continuity

	<b>Embodied</b>	<b>Symbolic</b>	<b>Formal</b>
<b>Subjective-social</b>	What does one see in the picture and how is the picture used in a mathematical discussion?	How are the formulas manipulated and how are the symbols used?	How does one understand, teach and use the definition?
<b>Objective</b>		$ g(x) - a $ $=  x^2 - 4 $ $=  (x + 2)(x - 2) $ $=  x + 2  x - 2 $ $\leq 5 x - 2 $	For every $\varepsilon > 0$ there is such a $\delta > 0$ that $ g(x) - g(a)  < \varepsilon$ for all $x$ satisfying $ x - a  < \delta$ .

This framework gives a viewpoint in which mathematical activity is an interplay between six dimensions of mathematics.

### 3 A continuous nowhere differentiable function and the six dimensions of mathematics

In this section, we present a novel construction of a continuous nowhere differentiable function and discuss our construction from the viewpoint of six dimensions of mathematics.

#### 3.1 Continuous nowhere differentiable functions

Continuous nowhere differentiable functions have an important role in the development of mathematics in the 19'th century. After the first discovery of a continuous nowhere differentiable function by Karl Weierstrass (1872), a great variety of constructions leading to such functions have been found (see e.g., Thim 2003). Being extremely counterintuitive such functions and their existence present also an interesting challenge for learning of the basic concepts of analysis and in mathematical thinking in general. For example, David Tall has been using a construction which he calls the Blancmange function in many of his writings (see e.g., Tall & Di Giacomo, 2000; Tall, 1982).

Such functions are related to first year analysis courses in university mathematics. Mostly their existence is only mentioned in analysis courses without going to details.

Our example of such a function is related to the use of pictures in communication mathematics. It seems that explicit reliance on Tall's embodied world is of special interest in connection to such technical mathematics. We shall present a new construction of a nowhere differentiable continuous function. We shall first discuss the construction of the function and the proofs of its special properties on the level of pictures. These pictures are not machine-made graphs of the function. Instead, they present the thinking behind the construction and therefore can be used as a basis of argumentation.

#### 3.2 A visual construction of the function $f$

We shall give a construction of a continuous nowhere differentiable function by visual means. The construction of our continuous nowhere differentiable function  $f$  and the discussion of its properties are written below so that the presentation suits for a group of students in a university course of analysis. Especially it is assumed that the students know in advance the basic properties of the real line including completeness and 'epsilon-delta'-definitions for continuity and differentiability.



Constructions of continuous nowhere differentiable functions usually rely on several theorems of analysis. The construction we present next is in this sense simpler. Besides the definitions of continuity and differentiability only a simple principle concerning nesting closed intervals will be used.

We shall consider the function  $f$  defined during the following imaginary discussion between a Teacher (T) and a Student (S). Originally the function will be defined for  $x$  satisfying  $0 \leq x \leq 1$ . Later a simple way of extending it to the whole real line is indicated.

In the discussion T and S look at the pictures appearing in [Figure 5](#). While the pictures as such belong to the objective side of Tall's embodied world, they are used on the subjective-social side of mathematics in the discussion.

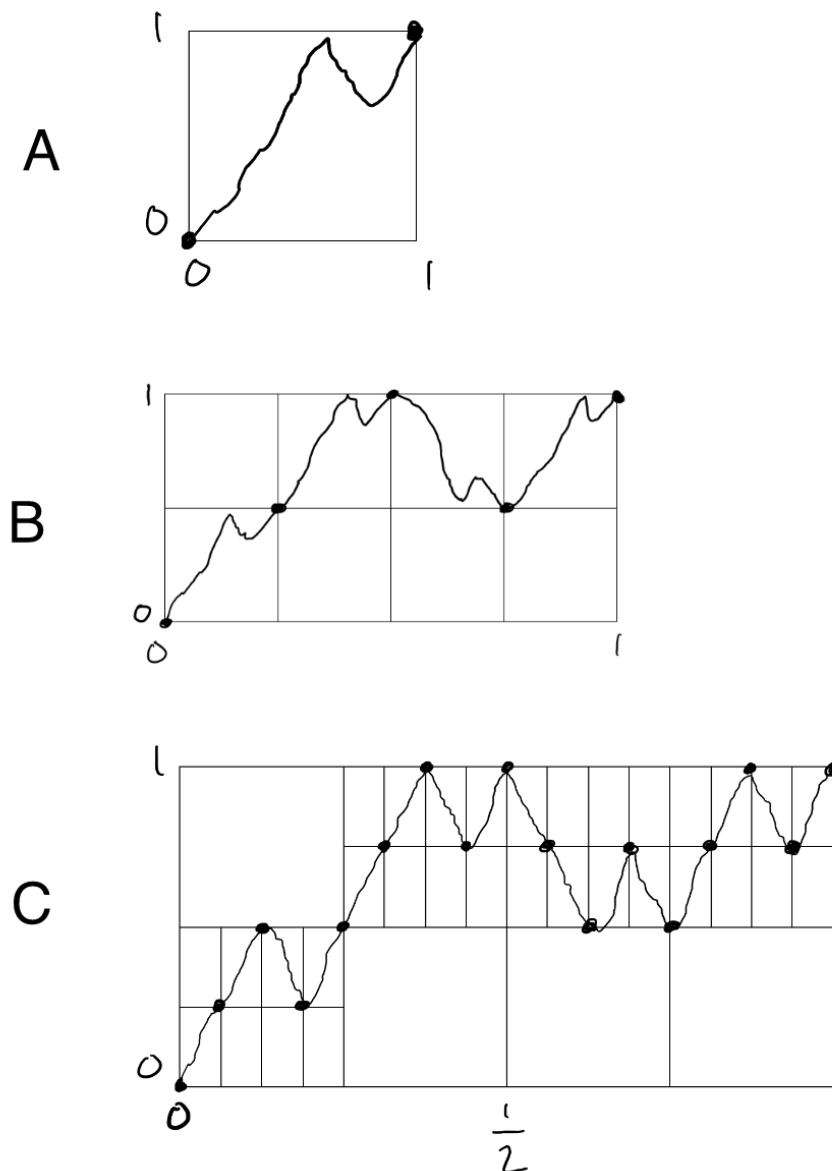


Figure 5. Explanation of the definition of  $f$

T: Let me show you a very interesting function

S: Fine! What is the definition?

T: Actually, I am not going to give a simple definition. Rather I use pictures to describe a process of adding information so that all the values will eventually be determined.

S: Exciting!

T: Look at picture A (of Fig. 5). We start with the information that  $f$  goes from the bottom left corner to the top right corner of the unit square.

This means that at the beginning we know that  $0 \leq f(x) \leq 1$  for  $0 \leq x \leq 1$ . Moreover,  $f(0) = 0$  and  $f(1) = 1$ . In picture A, a sketch of a graph is drawn only to give a feeling of what kind of a function we have in mind.

S: OK. But this does not tell much.

T: Look at picture B. At the next step we cut the square horizontally into four and vertically into two. This gives the smaller rectangles shown in the picture. And the function goes through some of these small rectangles as the sketch of a graph indicates.

So  $0 \leq f(x) \leq 1/2$  as  $0 \leq x \leq 1/4$ ;  $1/2 \leq f(x) \leq 1$  as  $1/4 \leq x \leq 2/4 (= 1/2)$ ;  $1/2 \leq f(x) \leq 1$  as  $2/4 \leq x \leq 3/4$  and  $1/2 \leq f(x) \leq 1$  as  $3/4 \leq x \leq 1$ . Moreover,  $f(0) = 0$ ,  $f(1/4) = 1/2$ ,  $f(2/4) = 1$ ,  $f(3/4) = 1/2$  and  $f(1) = 1$ .

S: The function seems to be in all these smaller rectangles somehow similar to the whole function in the original unit square with the exception of the third one.

T: Good! The third rectangle will be like the others, but everything is only upside down.

S: OK!

T: We know at this stage that we have rectangles in which the function goes from a left corner to the opposite right corner. To get more information we keep on cutting our rectangles to smaller. At each step we cut the rectangles horizontally in to four and vertically into two.

Look at picture C. There the next step / third step is drawn.

S: Yes, a similar idea seems really to repeat itself! But the small rectangles become all the time somehow different.

T: Can you say how they become different?

S: They become somehow more and more narrow!

T: indeed! Look at the ratio of the height to the width of these rectangles. Can you say what happens to it as our construction goes on?

S: It seems to increase all the time! The function becomes all the time somehow steeper and steeper.

To spare space, we end the dialogue and describe what happens. The above process is repeated infinitely many times. In cases where the function “goes from the upper left corner to the lower right corner” (which is above the case on the subinterval  $\left[\frac{1}{2}, \frac{3}{4}\right]$ ), the picture is used “upside down” as in [Figure 6](#).

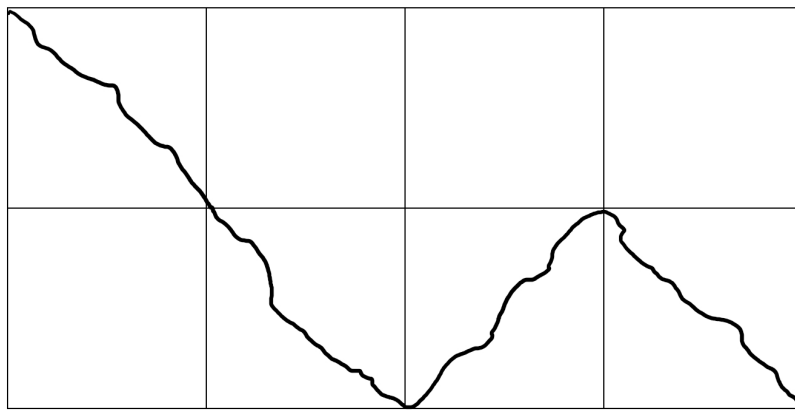


Figure 6. One more detail in the definition of  $f$

Pictures can be used also for communicating proofs for the continuity and nowhere differentiability of  $f$  – or at least for indicating the thinking behind the formal proofs.

To do this, some notation will help. Notice that at each step  $n$  of the construction we use rectangles with certain width  $w_n$  and height  $h_n$ . Indeed,

$$w_1 = 1 \text{ and } w_{n+1} = \frac{1}{4} w_n;$$

$$h_1 = 1 \text{ and } h_{n+1} = \frac{1}{2} h_n.$$

Especially, the form of these rectangles is characterized by the ratio

$$\frac{h_{n+1}}{w_{n+1}} = 2^n.$$

The first immediate consequence of the construction is that whenever

$$|x - t| < w_n,$$

the points  $(x, f(x))$  and  $(t, f(t))$  of the graph of  $f$  must lie in the same or consecutive rectangles. (If we were discussing such pictures in front of us, it would be natural to show with one's finger the points discussed. So, gestures appear naturally on the subjective-social side of mathematics.)

Thus

$$|f(x) - f(t)| < 2h_n.$$

It follows from this observation that  $f$  is uniformly continuous (see [Figure 5](#), picture C).

To prove the nowhere differentiability of  $f$ , we take a new look at the pictures used before and make a small addition to them. This is done in [Figure 7](#). To show that  $f$  is not differentiable at a certain point  $x_0$ , we shall consider the difference quotients

$$\frac{f(x) - f(x_0)}{x - x_0}$$

for certain other values  $x$ .

In every stage  $n$  of the construction, we can locate  $x_0$  in a picture like this. We can assume that  $f$  'goes' from the bottom left corner to the top right corner. (The other case where  $f$  'goes' from the top left corner to the bottom right corner is quite similar.)

Assume first that  $x_0$  is 'in' the rightmost quarter. Let the other value  $x$  in the difference quotient correspond to the left bottom corner. For geometric reasons we see that the absolute value

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

is at least the slope for the rising line drawn in the picture. Thus

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \geq \frac{1}{2} \cdot \frac{h_n}{w_n}.$$

But this ratio can be made as big as we like by choosing  $n$  big enough! Notice that if  $x_0$  is ‘in’ any other part of the picture, we have the same estimate. (If  $x_0$  is ‘in’ the leftmost part, then we take  $x$  to ‘correspond to’ the top right corner of the picture.)

This observation gives us the following result: For every  $x_0$ , every  $\varepsilon > 0$  and every  $M > 0$ , there is  $x$  for which  $|x - x_0| < \varepsilon$  and

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| > M.$$

Especially,  $f$  is nowhere differentiable since the difference quotient corresponding to any  $x_0$  cannot have a limit.

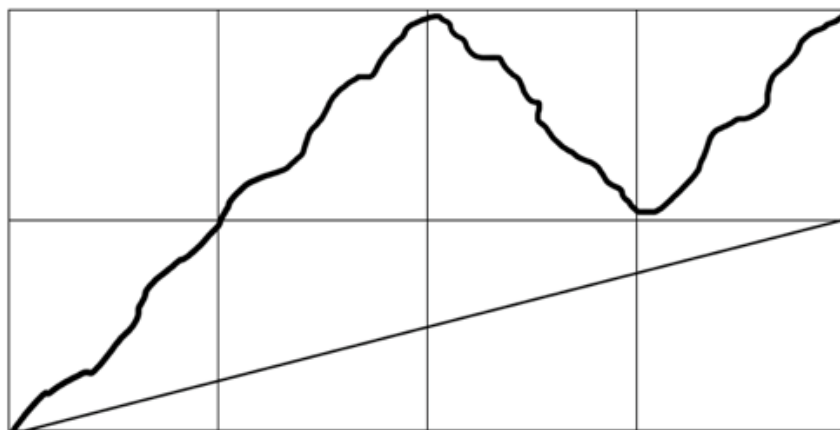


Figure 7. Why  $f$  is not differentiable

(The picture is of course too wide here, but it is meant to express the idea.)

One theoretical detail has been omitted so far. The reader may wonder how actually to prove that this construction leads to exact values  $f(x)$ . This follows from the simple principle that if we consider a nesting sequence of closed intervals  $[a_1, b_1], [a_2, b_2], [a_3, b_3], \dots$  where  $a_1 \leq a_2 \leq a_3 \leq \dots \leq b_3 \leq b_2 \leq b_1$  and where length  $b_n - a_n$  tends to 0 as  $n$  increases, then there is a unique number lying in all these intervals. Indeed, this number is the supremum of  $a_1, a_2, a_3, \dots$

This property is actually very interesting for several reasons. It is rather obvious when we ‘look at’ the real line. Hence it is very close to our ‘visual image’ of the real line. It is also rather easy to prove this property in an introductory course in analysis. Moreover, this property is a nice version of the compactness of closed intervals, and

it can be used to give a uniform way of proving the main consequences of compactness in an analysis course by ‘cutting closed intervals into to halves’.

In this construction and in the arguments above the authors like especially the feature that all the thinking is completely visual (or embodied in the pictures, as will be said later in this paper). It would be wonderful to present this at a blackboard!

More exactly, the visual proof consists of the above pictures and a discussion while observing the pictures. This will suffice to convince most novices and experts. In case we would like to write a formalized proof, we could use such a discussion as a recipe.

Since  $f$  is continuous and nowhere differentiable, also that the function  $g(x) = f(x) - x$  is continuous and nowhere differentiable. This function has the additional property that  $g(0) = g(1)$ . Therefore, extending  $g$  on the whole real line is especially simple: just put  $g(x) = g(x - n)$  when  $n \leq x < n + 1$ .

Finally, in [Figure 8](#) there is a ‘realistic’ picture of the function  $f$  produced by Maple using a code kindly written for us by Antti Rasila.

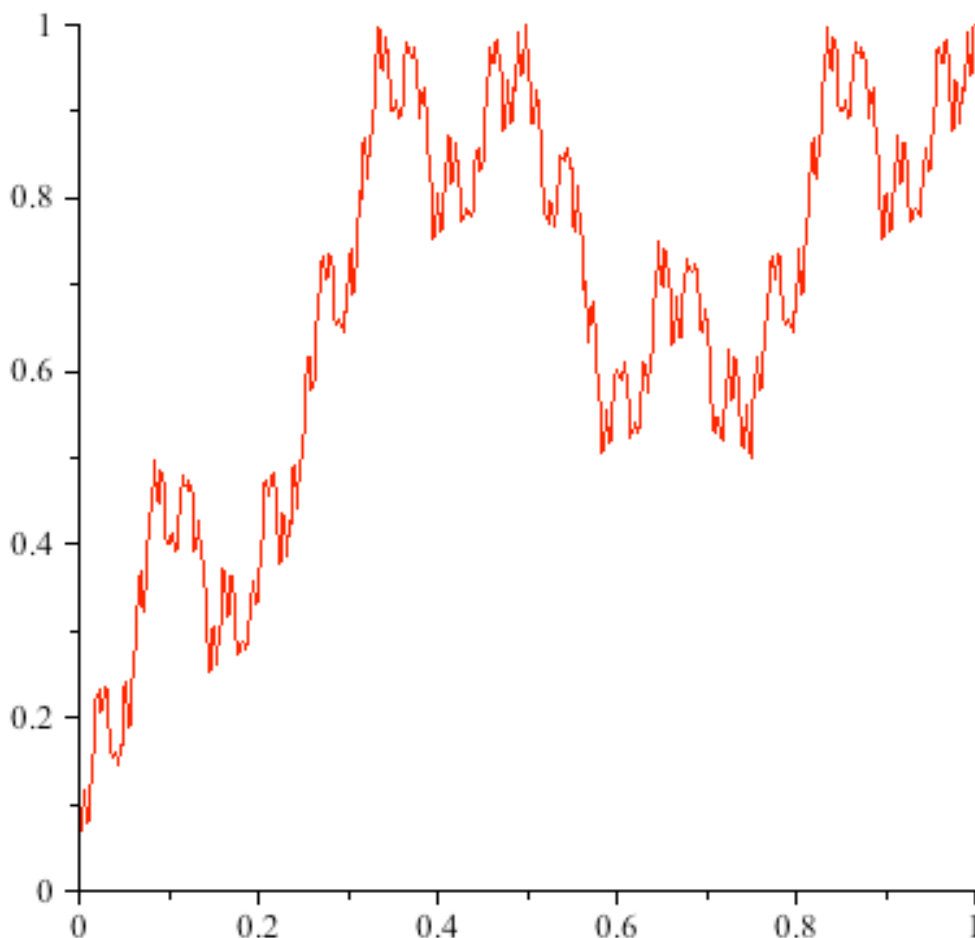


Figure 8. The portrait of the function  $f$



### 3.3 The function $f$ from the point of view of the six dimensions of mathematics

We defined in the previous section a continuous nowhere differentiable function using pictures and dialogue. This was very much like the dialogue used earlier in connection to the notion of continuity. In this section we shall relate this to our idea of  $2 \times 3 (= 6)$  dimensions of mathematics.

A continuous nowhere differentiable function is a very theoretical object. As such it seems to belong strongly to Tall's formal world. There are many constructions of such functions in literature, and they are presented usually in a theoretical way.

The function whose construction and properties were discussed in the previous section is essentially quite similar. But the main interest in the previous section was how to think and communicate about our function. This was done by means of sketchy pictures and a dialogue.

The dialogue and thinking were strongly subjective-social. The pictures as part of the communication had also a subjective-social role. It is probable that the participants of the dialogue constructed several mental images of their own related to the pictures and sayings (and gestures) of the other participants.

From the point of view of Tall's worlds, this happened in the embodied world. Hence our construction was presented in the dimension of the subjective-social side of the embodied world.

The pictures and a description of how to interpret them would belong to the objective side of Tall's embodied world when printed. The meaning of the pictures as such would have been very hard to understand without the dialogue or a good written explanation.

There was also an explanation of how to prove the continuity and nowhere differentiability of  $f$ . This used simple calculations on the proportions of the rectangles appearing in the construction. There we added aspects of the subjective-social side of Tall's symbolic world. And when printed, this addition was in the objective side of the symbolic world.

Finally, if we would have continued the discussion to the meaning and interest in continuous nowhere differentiable functions, we would have entered the subjective-social side of Tall's formal world. And when printed, this would have happened in the objective side.

So, all the  $2 \times 3$  dimensions of mathematics had a role in what was done.

## 4 Conclusion

We introduced a novel framework of six dimensions of mathematics by combining our view of two sides of mathematics with the framework of three worlds of mathematics. The idea of objective and subjective-social sides of mathematics does not as such appear in other distinctions presented in literature. However, the view of two sides of mathematics can be seen as an extension of many prior distinctions. Especially, the construction of the function  $f$  presented in this paper supports the view that our two sides of mathematics and Tall's three worlds of mathematics fit nicely together in a sense that they look at the same mathematical scenery from two 'orthogonal' directions. Both of our two sides correspond to aspects of most of Tall's three worlds and each of Tall's three worlds has aspects of both of our two sides. This holds even for the formal world for example in the sense that reading and making proofs belong to our subjective-social world.

In terms of developing university-level mathematics teaching, we considered two main examples. First, we presented a discussion on the definition of continuity which shared the expert's thinking with the students. Later, we analyzed this discussion using our theory of the six dimensions of mathematics. To the second we gave a construction of the function  $f$  presented in this paper and used it to give insight into the variety of mathematical thinking behind advanced mathematics. Tall's worlds can easily be seen as three steps of growth towards deeper and more abstract (expertise in) mathematics. But a more correct view seems to be that more than one of them are present in an expert's relation to mathematics. However, the secondary-tertiary transition includes a change in mathematical thinking as the formal world is emphasized at university (Tall, 2008). Therefore, explicit interplay between different worlds of mathematics is crucial in university-level teaching (cf. Oikkonen, 2009). One of the most interesting features of the construction and argumentation concerning the function  $f$  in this paper is that it serves as an example of an unusual route through the six dimensions of mathematics to present a piece of higher mathematics.

Regarding school mathematics teaching and teacher education, we also suggest more explicit interplay between different worlds of mathematics. Both Tall's three worlds and our two sides of mathematics are closely related to attempts to understand how mathematics can be made meaningful to people. Several studies show that use of multiple representations is crucial in teacher's profession (e.g., Dreher & Kuntze, 2015) and teacher education would benefit from more explicit links between

university mathematics and school mathematics (e.g., Hannula, 2018). Our framework is one viewpoint to develop such mathematical thinking both in pre-service and in-service teacher education.

Concerning further research, our framework can be utilized especially in analyzing mathematical discussion in authentic situations. For instance, the framework can be used in analyzing the elements of student groups' (un)successful problem-solving processes of in undergraduate mathematics courses. In addition, the framework gives a new lens to widely studied themes of representations and teacher knowledge (cf. Hannula, 2018). Therefore, the framework can be applied also in school mathematics and teacher education related research projects.

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# Enablers and obstacles in teaching and learning of mathematics: A systematic review in LUMAT journal

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This paper presents results of a systematic review of papers published at the LUMAT journal on the current issues positively and negatively affecting teaching and learning in mathematics, in concurrence with the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) guidelines. The analysis also offers insight into the most studied topics in mathematics education research, including key demographic and methodological characteristics such as year of publication, participants, education level, research methodologies, and research focus. Data was gathered from the studies published in the LUMAT: International Journal on Math, Science and Technology Education, starting from its first volume in 2013. So far, 225 articles were published in this journal, with 133 studies written in English and 51 studies related to mathematics. Although earlier studies support the notion that mathematics education is mostly traditional, this review suggests current research has thorough and positive outcomes, such that mathematics educators are likely to implement non-traditional approaches, encouraging student engagement, peer collaboration, and mathematical discourse. Certainly, in such learning environments, students tend to feel more motivated and less anxious about learning mathematics. They may also be more active and responsible in their learning, collaborate with peers, and get into mathematical discussions. Yet, there are also a number of difficulties and obstacles highlighted both in teaching and learning of mathematics. The findings might inspire several instructional implications for mathematics educators, curriculum developers, and researchers. Recommendations are given to add into what the existing literature claims and offer greater empirical evidence to support the verdicts.

Keywords: mathematics education, mathematics learning, mathematics teaching, systematic review, PRISMA

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## 1 Introduction

How do you remember being taught mathematics at school? If you were in school some decades ago, you could think of sitting in a row, watching the teacher quietly while s/he is solving a number of questions on the board, and then doing similar exercises (Rossi, 2015) until s/he thinks that the targeted learning outcomes are attained. How about nowadays? Is it still the same way? Definitely, with the increased knowledge of how students learn (Bransford, Brown, & Cocking, 2002), the recognition of ineffectiveness of traditional pedagogies (Hazari, Sonnert, Sadler, & Shanahan, 2010), and the availability of new educational technologies, this kind of





structured and teacher-centered approaches are not common nor desired practices in mathematics education (Milner-Bolotin, 2012). These days, learners are expected to be capable of more than applying arithmetic skills, but rather taking responsibility in learning process and possessing 21st century skills such as mathematical reasoning, critical thinking, and problem solving (Larmer, Mergendoller & Boss, 2015). Correspondingly, teachers are no longer expected to be transmitting knowledge, but rather acting as facilitator and engaging students in mathematical discourse structured around well-designed authentic activities (Markham, Lamer & Ravitz, 2006).

Especially, since the beginning of 1980s, problem solving has become an essential part of mathematics teaching and learning (Schoenfeld, 1985). According to Polya (1962), the term ‘problem solving’ refers to “finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable” (p. v). More clearly, Mayer and Wittrock (2006) explain it as “when you are faced with a problem and you are not aware of any obvious solution method, you must engage in a form of cognitive processing called problem solving. Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver” (p. 287). Here, it is important to note that the attribute “problem” is determined by the solver, more than the task itself, such that what might be a challenging problem for one solver can be just a routine exercise for other (Polya, 1962). It is commonly acknowledged that solving problems, especially open-ended problems through classroom discussion, helps students share strategies, insights, and observations with each other, engaging them in a quality mathematical discourse (Boud, Keogh & Walker, 1985), deepening their mathematical thinking, enhancing creativity (Pehkonen, 2001), and promoting diverse and flexible thinking, as well as positively influencing attitudes and self-efficacy in mathematics learning (Lester & Kehle, 2003).

More recently, project-based learning has taken the attention of educators and education researchers. In particular, it is, similar to problem solving, an active learning methodology which “engages students in learning knowledge and skills through an extended inquiry process structured around complex, authentic questions and carefully designed products and tasks” (Markham, Lamer & Ravitz, 2006, p. 4). It has been documented that when students learn mathematics through project-based learning, they are more capable of using mathematical knowledge in daily life situations (Drake & Long, 2009), remember the content longer (Wirkala & Kuhn,

2011), and have increased motivation toward learning mathematics (Larmer, Mergendoller & Boss, 2015). Beyond that, lately, in consent with unifying instruction, research on STEM (Science, Technology, Engineering and Mathematics) and STEAM (Science, Technology, Engineering, Art and Mathematics) education have spread comprehensively (Çiftçi, Topçu, & Foulk, 2020). Mainly, STEM and STEAM are interdisciplinary approaches that integrate the development of academic knowledge and skills beyond the specificities of the separate disciplines (Monkeviciene, Autukeviciene, Kaminskiene & Monkevicius, 2020). There is a growing body of research showing that incorporating mathematics education with other subject matters in a real, integral, and meaningful context evidently provides an effective platform for rich learning experiences (Brenneman, Lange & Nayfeld, 2019; García-Holgado, Camacho, & García-Peñalvo, 2019; Lawson, Cook, Dorn, & Pariso, 2018).

In brief, in the last decades, with the shift from structured and teacher centered pedagogies to advanced and learner-centric pedagogies, there was a significant transformation by what means mathematics was taught in schools. Yet, along with all this transformation, students still perceive mathematics as a difficult subject matter (Fritz, Haase & Rasanen, 2019) and still several students do not achieve well in mathematics (Sun, 2018). The literature highlights numerous challenges related to both inner and external conditions of a learner. In particular, the inner conditions include cognitive, affective, and motivational factors (Op't Eynde, De Corte & Verschaffel, 2006), such as lack of interest, poor motivation, negative attitudes about learning mathematics, and negative beliefs about their ability and potential (Walker, Smith & Hamidova, 2013). Likewise, the external conditions include teacher factors, such as teachers' poor experience, subject knowledge, qualification (Holzberger, Philipp, & Kunter, 2013), as well as negative attitudes, interest, motivation (King-Sears and Baker, 2014), and efficacy beliefs about teaching (Woodcock & Reupert, 2016). In addition, the external factors consist of contextual factors, including under-resourced and large sized classes (Chiwiye, 2013), and negative attitudes of family members, friends, and society (Boaler, 2015).

If, as Pólya (1962) stated, mathematics is about inquiry, reasoning, and understanding how things fit together, then what could be added into mathematics instruction to help students deepen their mathematical thinking and sense making? How can mathematics learning happen in a concrete and playful way? How can students experience mathematics through creating, designing, and connecting

mathematical ideas? Which learning environments can inspire students to build their own mathematical thinking?

This study aims to provide a portrait of research on mathematics education, highlighting multiple aspects of studies published in the LUMAT: International Journal on Math, Science and Technology Education, starting from its first volume in 2013 so far. Specifically, the study aims to answer the following research questions:

1. What are the most studied topics in mathematics education research?
2. What are the enablers and obstacles in mathematics teaching?
3. What are the enablers and obstacles in mathematics learning?

## 2 Methodology

In this study, a systematic review was conducted on all the published papers in the LUMAT: International Journal on Math, Science and Technology Education, starting from its first volume at 2013 to volume 9 no 2 in 2021. While traditional literature reviews provide a review of knowledge on a general topic without applying a scientific methodology, a systematic review implies “a complete, objective and reproducible” (Linares-Espinós et al., 2018, p.502) synthesis of a clearly defined topic to answer particular research questions in a transparent (Gough, Oliver & Thomas, 2012), standardized, and systematic way (Higgins et al., 2021). Systematic review consists of identifying, selecting, analyzing, and synthesizing information derived from published studies, with explicit inclusion and exclusion criteria (Møller & Myles, 2016). To ensure credibility, consistency, and transparency, the researcher followed the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) guidelines and the four-phase flow diagram, addressing all the sections of a systematic review (Moher, Liberati, Tetzlaff, & Altman, 2009).

### 2.1 Eligibility criteria

The eligible studies in this review consist of all the articles published in this journal, since its first volume, including studies written in English and related to mathematics learning and teaching. So far, 225 articles were published in this journal, with 133 studies written in English (59.1%) and 51 studies related to mathematics field (22.7%).

## 2.2 Information sources

In June 2021, the researcher visited the website of the LUMAT: International Journal on Math, Science and Technology Education and accessed all the journal's papers in the archives, as it is an open access journal.

## 2.3 Search and study selection

The search was done manually by the researcher, as the search option in the journal website was filtering the studies only by title or author. The researcher examined all the papers volume by volume, first identifying the publications in English, then screening the papers by the title, abstract, and keywords for the limiters 'math', 'mathematics education', 'mathematics learning' and 'mathematics teaching'. [Figure 1](#) illustrates the flow of the study selection process, including identification, screening, eligibility, and included studies. The retained papers are marked with an asterisk in the references list and summarized in the Appendix.

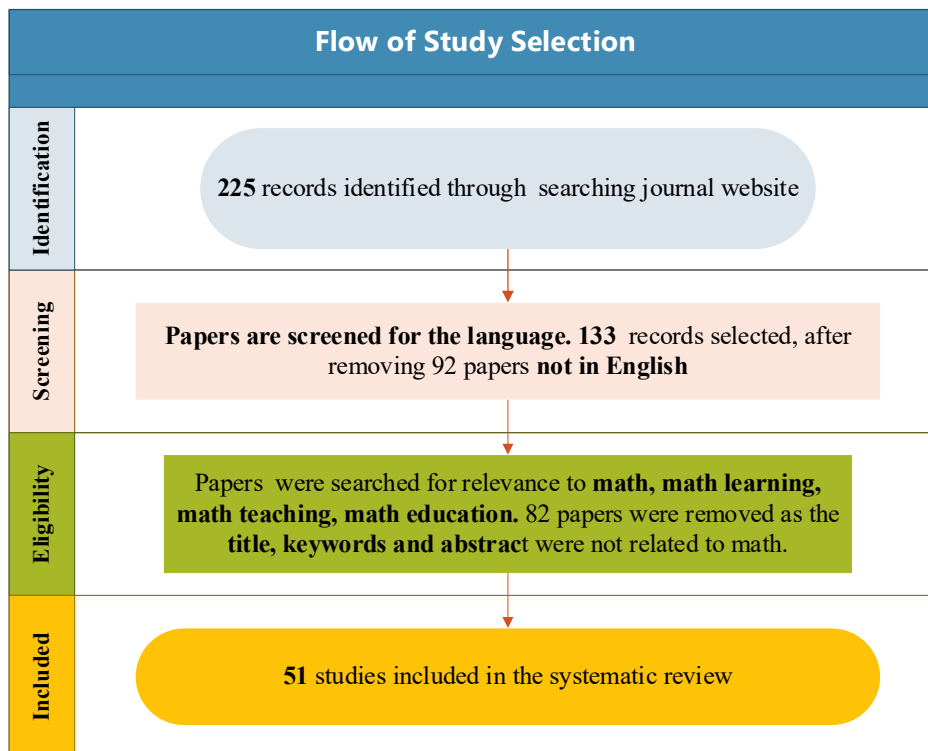


Figure 1. Flow of the study selection

## 2.4 Data extraction process

Data were extracted from the published papers about publication year, participants, education level, research methods, and research focus. Prior to extracting the data, the researcher established a coding protocol to analyze the data systematically. The variables and the codes are listed in [Table 1](#).

**Table 1.** Data items and codes

Year of Publication	Participants	Level of Education	Research Method	Research Focus
1. 2013	1. K-12 students	1. Higher education	1. Qualitative (e.g. case study, historical study, grounded study)	1. Math teachers and teaching
2. 2014	2. Undergraduate students	2. Secondary education (grades 10-12)	2. Quantitative (e.g. survey, experimental, correlational)	2. Math learners and learning
3. 2015	3. Graduate students	3. Primary education (KG, grades 1-9)	3. Mixed (e.g. explanatory, exploratory, multiphase)	3. Policy and curriculum
4. 2016	4. K-12 teachers	4. All	4. Conceptual (e.g. systematic review, reflection paper, opinion paper)	4. STEM education
5. 2017	5. Pre-service teachers	5. Others	5. Others	5. Others
6. 2018	6. Faculty members			
7. 2019	7. Parents			
8. 2020	8. Principles			
9. 2021	9. Others			

## 3 Results and Discussion

### 3.1 What are the most studied topics in mathematics education research?

The analysis of most studied topics in mathematics education included examining key demographic and methodological characteristics, including year of publication participants, level of education, research method, and research focus. With regard to publication years ([Figure 2](#)), the results reveal that starting from the year 2013 so far, a total of 225 articles were published in this journal (in 2013  $n=43$ , 19.1%, in 2014  $n=25$ , 11.1%, in 2015  $n=78$ , 34.7%, in 2016  $n=6$ , 2.7%, in 2017  $n=4$ , 1.8%, in 2018  $n=15$ , 6.7%, in 2019  $n=22$ , 9.8%, in 2020  $n=14$ , 6.2%, and in 2021  $n=18$ , 8%), with the highest number of publications in the year of 2015. In particular, 133 studies (59.1%) were written in English (in 2013  $n=20$ , 15%, in 2014  $n=2$ , 1.5%, in 2015  $n=45$ , 33.8%, in 2016  $n=3$ , 2.3%, in 2017  $n=3$ , 2.3%, in 2018  $n=12$ , 9%, in 2019  $n=21$ , 15.8%, in 2020  $n=13$ , 9.8%, and in 2021  $n=14$ , 10.5%), with the highest number of publications

in the year of 2015 as well. Among these studies, 51 studies (38.4%) were related to mathematics (in 2013  $n=2$ , 3.9%, in 2015  $n=13$ , 25.5%, in 2017  $n=1$ , 1.9%, in 2018  $n=5$ , 9.8%, in 2019  $n=14$ , 27.5%, in 2020  $n=9$ , 17.6%, and in 2021  $n=7$ , 13.7%), with the highest number of publications in the year of 2019, and no publication in the years of 2014 and 2016.

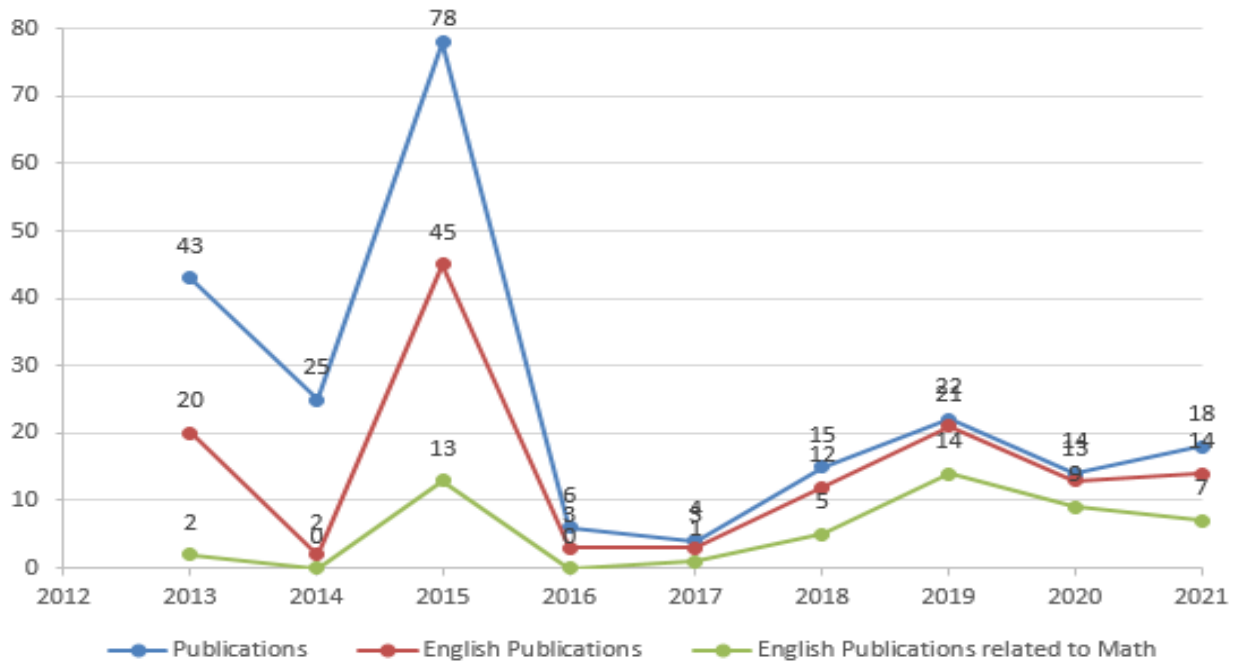


Figure 2. Publication Year

Regarding participants (Figure 3), in most of the studies the data was gathered from teachers ( $n=28$ , 54.9%), in particular from K-12 teachers ( $n=15$ , 29.4%), pre-service teachers ( $n=10$ , 19.6%), and faculty members ( $n=3$ , 5.9%). In addition, information was also collected from students ( $n=19$ , 37.2%), specifically from K-12 students ( $n=15$ , 29.4%), and undergraduate students ( $n=4$ , 7.8%), with no focus on graduate students. In few cases, information were extracted from published materials ( $n=8$ , 15.7%), as well as principles ( $n=2$ , 3.9%) and parents ( $n=1$ , 1.9%).



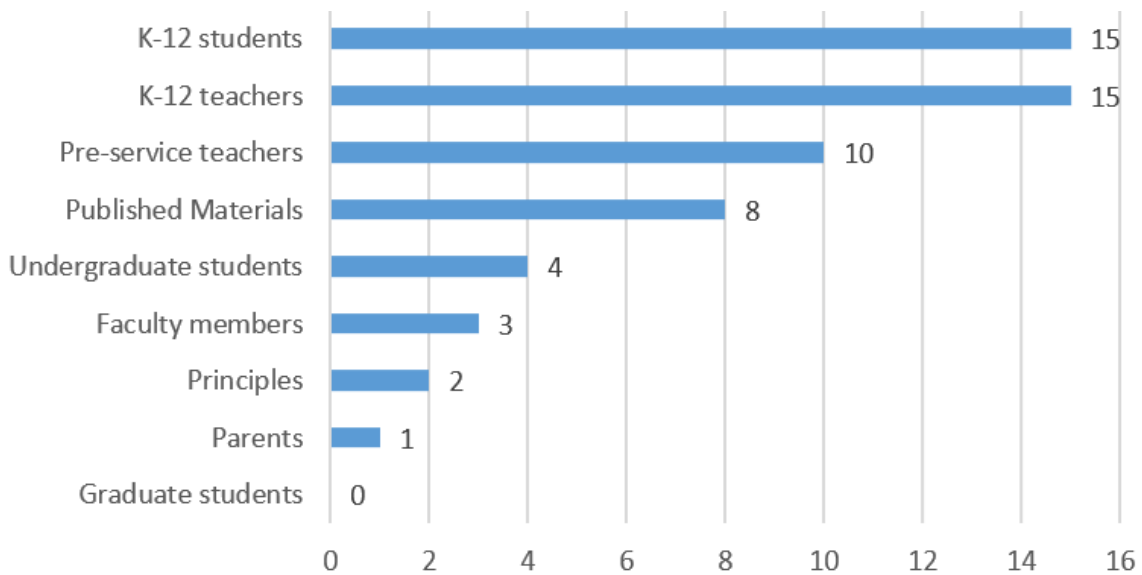


Figure 3. Participants

Concerning level of education, the results show that in most of the studies a strong focus was on primary education which includes kindergarten and grades 1 to 9 ( $n=23$ , 45.1%) followed by higher education ( $n=15$ , 29.4%) and secondary education including grades 10 to 12 ( $n=10$ , 19.6%). Moreover, a few studies included all levels of education ( $n=6$ , 11.8%).

In terms of research methods, qualitative analysis was the most widely- used research methodology ( $n=22$ , 43.1%), followed by mixed ( $n=13$ , 25.5%), quantitative ( $n=9$ , 17.6%), and conceptual analysis ( $n=7$ , 13.7%). Lastly, in terms of research focus, in most of the studies the focus was highly on mathematics teachers and teaching ( $n=25$ , 49%), followed by mathematics learners and learning ( $n=20$ , 39.2%), and STEM education ( $n=5$ , 9.8%), with no emphasis on educational policy or curriculum.

### 3.2 What are the enablers and obstacles in mathematics teaching?

Across the fifty-one studies analyzed here, half of the research on mathematics education were related to internal and external factors positively or negatively affecting mathematics teaching. Overall, one of the main overarching themes identified from this analysis was that well-designed professional development activities and quality teacher training programs have a great impact on teachers' and pre-service teachers' knowledge, competence, and self-efficacy in teaching. Mostly, when there is a mutual trust among teachers and experts, a good teacher-expert collaboration and quality discussions, teachers tend to form a habit of personal reflection on their professional learning and look for solutions to make changes in

their mathematics teaching (Namsone, Čakāne & France, 2015).

In this aspect, a study by Kuzle (2019) and Hannula (2019) show that after receiving a professional training about problem solving, pre-service teachers evidently improved their mathematical content knowledge and problem-solving competence. Likewise, a study conducted by Heikkinen, Hästö, Kangas and Leinonen (2015) reveal that after a one-day, on-site professional training event, most of the teachers started questioning and challenging their attitudes towards mathematics teaching. On the other hand, some teachers were still resilient to change their traditional teaching methods due to external factors such as “lack of time, equipment and ready-to-use materials” and “lack of colleague support who share their vision” (p. 914). In addition, there were a number of internal factors, such as “lack of self-confidence in changing teaching methods” and “fear of failure as a teacher” (p. 915). In like manner, a study conducted by Wadanambi and Leung (2019) suggest that “contextual factors such as examination-oriented expectations, time constraints and previous learning experiences” have a significant impact on teachers’ actual teaching practices. In this aspect, it is highly important that teacher education programs focus not only on enhancing teachers’ instructional preparedness but also on preparing them affectively with high levels of efficacy and confidence in teaching (Ekstam, Linnanmäki & Aunio, 2017).

Another interesting outcome of this review is that problem solving appears to be one of the most commonly used method for teaching mathematics. Such as, a study by Koponen (2015) support the proposition that although implementing problem solving might be challenging and time consuming, it is an essential part of developing students’ mathematical thinking and problem solving skills. Especially, for an effective problem solving experience, it is highly recommended that instructors select problems for clear and pre-determined goals, ask students share their point of views with each other, and provide them appropriate guidance while working on finding the proper solution. Here, it is worth noting that while providing guidance, the type, number, and quality of teacher guidance have a great impact on students’ problem solutions (Kojo, Laine & Näveri, 2018). In particular, if a teacher provides too much help or reveals the solution, this ruins the problem solution process and turns an original problem into a standard task. Hence, research suggest that teachers learn how to properly guide their students with variety of probing and guiding questions. For example, teachers can ask probing questions (e.g. How did you solve this?) to lead students to explain their mathematical thinking and ideas. Next, they can ask guiding

questions (e.g. Why do you think it is not valid?) to leads students to think about the problem in a different way or to justify their solution. Moreover, teachers can ask factual questions (e.g. How many solutions have you found?) to motivate students to progress in their thinking process. Likewise, a study conducted by Luoto (2020) show that when a teacher balances between dialogic and authoritative speech, giving all students equitable chances of practice, students get active and participate in more productive mathematical discourse. However, when a teacher holds on authoritative approach, seeing discussions as useless and believing that students need strict procedural guidance, students have very limited classroom discourse where the participation happens mostly with short answers.

Similarly, in a study on problem solving with a STEM/STEAM focus, White and Delaney (2021) found that when teachers implement real-world project-based or problem-based learning, where students are in the center of their learning and learn by doing, students achieve higher learning outcomes, and develop positive attitude towards science and mathematics learning. As for research on problem solving, the findings are relatively straightforward, highlighting the importance of problem solving in enhancing students' mathematics learning; it becomes highly important for mathematics educators to have a clear understanding on how they can enhance students' problem-solving proficiency. In light of this, Chapman's (2015) study suggests that it requires more than knowing how to solve a problem. Particularly, in addition to being proficient in problem solving, mathematics educators need to understand "what a student knows, can do, and is disposed to do" and "how and what it means to help students to become better problem solvers", as well as "nature and impact of productive and unproductive affective factors and beliefs" (Chapman, 2015, p.31).

Finally, in addition to the above mentioned aspects, the results of this systematic analysis suggest that, in most of the studies analyzed here, mathematics educators were likely to possess positive characteristics, such as being "a life-long learner, patient, soft, friendly, calm, joyful, self-confident, knowing, and able to withstand hard use when needed" (Portaankorva-Koivisto & Grevholm, 2019, p.107). In addition, in most of the studies, mathematics teachers were reported to be implementing non-traditional teaching approaches including cooperative learning, deductive approach, inductive approach, and integrative approach. According to a study conducted by Cardino and Ortega-Dela (2020), mathematics teachers were mostly applying cooperative learning, followed by demonstration and repetitive

exercise. The researchers suggested teachers to use think-pair-share, round table activities, and jigsaw discussions for enhancing cooperative learning in mathematics education. For inductive approach, they suggested using observation, generalization, testing, and verification. Furthermore, for integrative teaching, the researchers advised mathematics teachers to foster students' creativity, use age appropriate materials, and generate new interdisciplinary ways for presenting old topics.

### 3.3 What are the enablers and obstacles in mathematics learning?

Regarding research on mathematics learning, the analyzed studies can be broadly consolidated into four key aspects, as learning mathematics via technology, impact of cognitive and affective variables on mathematics learning, influence of non-traditional pedagogies on mathematical discourse, and issues related to STEM learning. To start with, as regards to usage of technology in mathematics learning, in a study, Milner-Bolotin, Fisher and MacDonald (2013) examined the implementation of technology-enhanced pedagogy in different learning settings and suggested that classroom response systems (clickers) evidently improve student engagement, reduce anxiety, and enhance students' conceptual understanding in mathematics. Next, a study by Kuzle (2015a), on what learners could gain while working on geometry with a dynamic geometry software, revealed that with the use of software students could go beyond memorization. Indeed, they engaged in solving a wide variety of open-ended problems, which helped them apply the theoretical facts into practical situations and increase their mathematical understanding. In a further study, Kuzle (2015b) examined problem solver's cognitive and metacognitive behaviors while using the same dynamic geometry software. The findings suggest that the use of software supported the learner to engage in a variety of cognitive and metacognitive behaviors, such as gathering information, exploring, conjecturing, generating precise visual inputs, and finding possible solutions. In addition, the feedback provided by the software was assisted the problem solver to make effective decisions and actions. Lastly, in a study, Kaarakka, Helkala, Valmari and Joutsenlahti (2019) examined the impact of an online tool, called MathCheck, on students' level of conceptual understanding. Briefly, what made this tool potent was that it was checking the problem solution step by step and providing detailed feedback to the problem solver, more than an incorrect/correct verdict. Overall, the findings support the proposition that using technology helps students in independent studying and enhance a deeper conceptual understanding in mathematics learning.

Next, regarding research on cognitive and affective aspects, a study by Nyman and Sumpter (2019) revealed that students possess both intrinsic and extrinsic motivation for learning mathematics. Indeed, there is a high association between students' intrinsic and extrinsic motivation such that they are intertwined and hard to separate. In another study, Viholainen, Tossavainen, Viitala, and Johansson (2019) examined the challenges students face with respect to mathematics proof and proving. The results show that even though students were highly motivated to learn proof and proving, there were a number of factors hindering their proving skills, such as fear of making mistakes, lack of experience, low self-efficacy, and lack of knowledge about mathematical content. In this aspect, a study by Viitala (2015) suggest that even though in the existing education system students' educational motivation, positive self-image and self-confidence do not have an influence on their mathematics grades, it is essential that educators consider "taking responsibility of own learning, expressing mathematical thinking and applying mathematics in different environments" as a part of the assessment criteria (p. 148).

As for research on implementation of non-traditional pedagogies in mathematics learning, in a study, Rossi (2015) examined the impact of constructive teaching and technology on mathematics learning, and found that with non-traditional approaches it is possible to challenge and change students' poor engagement and negative attitudes towards mathematics learning. In a similar vein, Ambrus and Barcsi-Veres (2015) investigated traditional versus student-centered learning environments, and found that working in groups on open-ended math problems enhance collaboration and communication among students as well as improving their problem solving skills. In particular, when students worked in teams, they used more mathematical language to explain their ideas to each other. Specially, slow learners had more time to understand the given task and participated more actively in the problem solving process. However, in spite of being an effective learning tool, cooperative learning was reported to be time-consuming, causing a noisy environment, and disruptive for students who prefer to work alone. With a similar context, Viro and Joutsenlahti (2020) investigated the impact of project-based learning on students' level of mathematics attainment, and proposed that problem-based learning significantly improves students' grades in mathematics. Yet, the researcher also pointed out that the group formation is a critical issue in problem-based learning setting such that "a hard-working group can support and inspire a pupil to work and learn more, but on the other hand, a strong group may encourage a pupil to be a passenger" (p. 129). In



addition, when a student has most of his/her group members a lot weak, s/he may feel them as burden in his or her learning. Indeed, in an interesting research, Cardino and Ortega-Dela (2020) examined how students' learning styles influence their academic performance, and found that most of the students had a combination of dependent, collaborative and independent learning styles, and among these learning styles, although most of the students were collaborative, only independent learning style had a significant impact on improving academic performance.

In a study, Mason (2015) suggest that being stuck is a math problem could enhance learning about mathematics and mathematical thinking, as it opens the ways for inspiration and mindfulness. In this aspect, a study conducted by Laine, Ahtee, Näveri, Pehkonen, and Hannula (2018) focused on how students' mathematics learning was challenged when their teachers requested them to write down their thinking while solving problems. Based on the results, it was evident that deep questioning activated students' mathematical thinking, especially writing about their own thinking helped them "to remember and confirm new mathematical understanding" (p. 102). Hence, the researchers suggested that well-designed problem solving tasks, games, and class discussions are of high importance as they create a motivating context for learning and promote sharing of ideas, making sense and reasoning. Certainly, a study by Mononen and Aunio (2013) also suggest that when learners solve more problems and get more acquainted with mathematic topics, they performed better in exams, especially in problems related to numbers, listing, and arithmetic. Furthermore, from a different aspect, Alfaro Viquez and Joutsenlahti (2020) examined the impact of languaging exercises on promoting understanding in mathematics. Particularly, during the languaging exercises, students were given opportunities to participate in the construction of their knowledge by using a combination of symbolic, natural, and pictorial languages. The results showed that using different languages enhanced "the acquisition of skills necessary to be mathematically proficient and are a useful tool for revealing students' mathematical thinking and misconceptions" (p. 229). Likewise, a study by Luoto (2020) suggest that students' participation in mathematics discourse improve their mathematics learning.

As a final overarching aspect, in terms of research on STEM learning, Tomperi et al. (2020) investigated factors affecting students' attitudes towards learning mathematics and science. The results indicated that although female students realized the importance of science and mathematics for their future, male students were more



interested in career opportunities in the industry. Another interesting finding was that students who experienced innovative and student-centered teaching approaches were more motivated and less anxious about learning science and mathematics. Moreover, in a study, Cabello, Martinez, Armijo, and Maldonado (2021) investigated possible strengths and weaknesses of STEAM learning. Briefly, the study highlighted the strengths as promoting students' interests, engagement and motivation for learning processes. Especially, working with diverse materials and having enriched experiences mostly supported by technology allowed students to view scientific work as an exciting endeavor. On the other hand, the main difficulty was reported as teachers' management of student emotions and behavior, suggesting that "keeping class time within the average attention span and closely monitoring of children's fatigue may help prevent episodes of disruptive behavior" (p. 50). Lastly, a study by Milner-Bolotin and Marotto (2018) focused on the effect of parental engagement on students' STEM learning. The results highlighted the fact that although parents have a positive impact on students' STEM engagement and achievement, due limited STEM knowledge and language issues, some parents have difficult time in supporting their children in their STEM journey. In this manner, "creating family-oriented STEM resources" and offering "school-related projects, homework assignments, out-of-school science clubs and visits to science centres" are found to be assisting and motivating parents to engage in STEM learning with their children. (p. 53). In a follow up study, Marotto and Milner-Bolotin (2018) also suggested that "school-family, parent-teacher, and parent-child interactions" are important networks of communication in promoting STEM education (p. 81).

#### **4 Conclusion and Suggestions**

As a portrait of research on mathematics education, this systematic review highlights multiple aspects of studies published in the LUMAT: International Journal on Math, Science and Technology Education, starting from its first volume in 2013 so far. Searches were rigorously conducted on 225 existing studies and 51 articles were identified, analyzed and synthesized to examine issues positively or negatively affecting teaching and learning in mathematics. Moreover, the analysis offers insight into the most studied topics in mathematics education research, including key demographic and methodological characteristics such as year of publication, participants, level of education, research methodologies, and focus.

Briefly, in terms of the most studied topics, the results show that although in the year of 2015 the journal had the highest number of publications, the studies related to mathematics education was mostly published in 2019. In respect of participants, data were mostly extracted from teachers, followed by students, with very few information gathered from principles and parents. Regarding level of education, several studies were related to primary mathematics education and higher education, yet studies at kindergarten level was very limited. Moreover, as regards of research methodologies, qualitative analysis was the most widely-used research method, followed by mixed, quantitative and conceptual analysis. Furthermore, in terms of research focus, most of the studies focused on mathematics teachers and teaching, followed by mathematics learners and learning, however no special emphasis was given to educational policies or mathematics curriculum.

Regarding issues affecting mathematics teaching and learning, the results reveal that most of the existing studies have thorough and positive outcomes. In brief, research on mathematics teaching supports the notion that well-designed and good quality professional development programs positively influence teachers' and pre-service teachers' knowledge, competence, and self-efficacy in teaching. Interestingly, in many studies, mathematics educators were reported to be implementing non-traditional approaches in their teaching, including cooperative learning, problem-based learning, project-based learning, and experiential learning. In particular, teachers were paying attention to encouraging student engagement, peer collaboration, and mathematical discourse. Certainly, in such positive and inspiring learning environments, students were found to be more motivated and less anxious about learning mathematics. Indeed, they were active and responsible in constructing their mathematical understanding, such that they were collaborating with their peers, using mathematical language to share their ideas, and getting into discussions and deep questioning. In that aspect, the results give signals that having a student-centered approach in mathematics education with open-ended problems, think-pair-share activities, Socratic questioning, as well as interactive games, online tools, and challenging tasks that include observation, testing and verification could be of high importance in promoting students' mathematical thinking, conceptual understanding, and academic performance. This claim can be validated with further empirical investigation and studies using experimental designs to inspire the instructional implications for mathematics educators, curriculum developers, and researchers.

Definitely, there are also a number of difficulties and obstacles highlighted in the existing research in mathematics teaching and learning. For instance, although commonly non-traditional teaching approaches were reported to be having positive impact on student learning, they were also found to be causing a noisy environment, being time-consuming and disruptive especially for students who prefer to work independently. As for collaborative work, results suggest that while some group formations were supportive and inspirational in nature, some others were highly unproductive, turning even hard working students into passive observers. Here, research underlines the importance of the type, number, and quality of teacher guidance in such learning contexts, where too much help ruins the problem solution process, making an original problem just a standard exercise. Equally, it is important to consider teachers' monitoring and management abilities, to keep the class time within student attention span and prevent disruptive behavior. Indeed, not every teacher is a fan of student-centric pedagogies. In particular, research show that some teachers were resilient to change their teaching approaches as they had limiting exam-oriented expectations, time constraints and previous learning experiences. Next, some teachers stated having lack of self-confidence in making a change and fear of failure as a teacher. In that aspect, it is possible to recommend that teacher education programs and in-service activities should not only emphasize a number of theoretical aspects on instructional preparedness but also enhance educators' practical experiences and develop self-efficacy and confidence in teaching.

## **5 Limitations and Implications for Future Studies**

This systematic review is limited to the analysis of the papers published in this journal. A broader dataset in terms of number of journals, language, and research context would greatly improve the understanding of enablers and challenges in mathematics learning and hence support the development of mathematics education. Certainly, what is gathered in one study may not be the same or similar to what is gathered in other, as every research endeavor has its own characteristics. In order to add to what the existing literature claims and offer greater empirical evidence to support the verdicts, further studies can be conducted in mathematics education particularly by means of different settings and characteristics.

Furthermore, while this review provides insights into what exists in the current literature, future studies can focus on specific issues to deepen our understanding of mathematics education. For instance, more research is needed on affective variables

such as teachers' and students' educational motivation, attitudes and self-confidence, as well as the influence of parents' and friends' attitudes toward mathematics learning. Next, research can be conducted on the relatively unexplored field of high-performing or gifted students in mathematics, such as examining how to enrich the current educational materials and expand on the standard goals so high achievers and gifted learners get more opportunities to benefit from the formal mathematics curriculum. Furthermore, as research underlines the fact that traditional examinations are not correct ways for measuring the twenty-first-century skills (Bell, 2010), more research is needed to investigate the validity of authentic assessments, such as portfolios, self-assessment, team work, and peer evaluations (Erdogan & Bozeman, 2015). Finally, whilst research on STEM and STEAM education is still in its infancy, further studies can be conducted to gather a more nuanced understanding of how to integrate these disciplines in mathematics education, especially starting in the early years.

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# Rudimentary stages of the mathematical thinking and proficiency: Mathematical skills of low-performing pupils at the beginning of the first grade

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A national-level dataset ( $n = 7770$ ) at grade 1 of primary school is re-analyzed to study preconditions in proficiency in mathematical concepts, operations and mathematical abstractions and thinking. The focus is on those pupils whose preconditions are so low that they are below the first measurable level of proficiency in the common framework with reference to mathematics (CFM). At the beginning of school, these pupils may not be familiar with, e.g., the concepts of numbers 1–10, they may not be aware of the consecutive nature of numbers, and they have no or very limited understanding of the basic concepts of length, mass, volume, and time. A somewhat surprising finding is that the key factor explaining the absolute low proficiency in mathematics appeared to be a low proficiency in listening comprehension. This variable alone explains 41% of the probability of belonging to the group of pupils who are not able to show proficiency enough to reach the lowest level in any of the criteria. It is understandable that, if language skills are underdeveloped in general, a child is not expected to master the specific mathematical vocabulary either and, hence, the low score in a test of preconceptions in mathematics too. Other variables predicting the absolute low level or preconditions of mathematics are the decision on intensified or special support, status of Finnish or Swedish as second language, and negative attitudes toward mathematics.

Keywords: mathematical thinking, mathematically low-achieving students, national assessment in mathematics, pre-primary education, primary education

## 1 Introduction

Mathematical competence is one of the key skills needed in modern society. From the viewpoint of socializing citizens to mathematical concepts and operations, as well as abstraction and thinking, teachers in schools are the key persons because pure mathematics is rarely a natural hobby of children, unlike sports, handicrafts, or reading. The main contents of mathematics are learnt, practically speaking, exclusively in or through the school: in the first grades, in mathematics lessons and while doing school homework (Metsämuuronen, 2013a). From this viewpoint, measuring the level of mathematical thinking and proficiency in mathematics in general makes sense at higher grades. Usually, in Finland, national assessments of learning outcomes are administered at grade 9 (Metsämuuronen, 2009) and the

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international PISA (Programme of International Student Assessment) and TIMSS (Trends in International Mathematics and Science Study) comparisons at grade 8 (e.g., PISA, 2019; TIMSS, 2020) or even later in adulthood (see the Programme for International Assessment of Adult Competencies [PIAAC], OECD, 2016). Hence, the end-product of the socializing the citizens in mathematics during the school years is well-known and well-followed-up.

Although the systematic socialization to mathematics happens mainly in and through school, children have learned a lot matters that are related to mathematics even before the school age—7 years in Finland. These preconditions on mathematics are in the focus of this article. At the national level, it is very rare to see measures of mathematical thinking and competencies at the beginning of schooling, that is, large studies on what are the first stages of development of mathematical thinking and what kind of proficiency are largely lacking (see, however, e.g., Lerkkanen et al. 2012, where mathematics skill was assessed as a part of the First Steps study by the fluency in counting forwards and backwards number sequences). In 2018, the Finnish National Education Evaluation Centre (FINEEC) launched a longitudinal assessment to measure the achievement level of pupils and students at different stages of their school years in mathematics and mother language. The first measurement was administered in the first weeks at grade 1 with a minimal effect of school in mathematical thinking (see methods in Metsämuuronen & Ukkola, 2019 and results in Ukkola & Metsämuuronen, 2019; Ukkola, Metsämuuronen, & Paananen, 2020). The dataset gives quite a unique possibility to study the outcome of early childhood development from the mathematic development viewpoint.

In this article, this unique dataset of preconditions of mathematics at grade 1 ( $n = 7770$ ) is re-analyzed from the viewpoint of mathematical thinking by using the common framework with reference to mathematics (CFM) suggested by Metsämuuronen (2018). CFM divides the mathematics skills into three criteria: proficiency in mathematical concepts, proficiency in mathematical operations, and proficiency in mathematical abstractions and thinking. In Section 2, the factors affecting the development of mathematical skills in the early childhood are discussed which is followed by discussing the characteristics of CFM in Section 3 and methodological matters for the empirical section in Section 4. Section 5 combines these and presents results of proficiency of mathematical concepts, procedures and thinking at the beginning of the grade 1 in schools in Finland. The focus is, specifically, in predicting and detecting the children in whom mathematical skills are



underdeveloped or whose skill level is lower than the measurable level when they are at the age of starting school, that is, when they 7 years old. The main research question is, what variables characterize the pupils whose mathematical skills and thinking are very low—even below a measurable level, and which background factors could be used to detect such children. The main research question is divided to three sub-questions:

1. What kind is the overall distribution of preconditions in mathematics at the beginning of the first grade?
2. How do personal factors characterize the pupils with very low preconditions in mathematics at the beginning of the first grade?
3. How do family factors characterize the pupils with very low preconditions in mathematics at the beginning of the first grade?

## 2 Some known factors affecting the development of mathematical thinking in the early childhood

In comparison with many other countries, in Finland the children enter the school rather late, typically when they turn 7. Because the first 6 years may be radically different, children enter the school with a wide variety of mathematical skills (see Metsämuuronen, 2010; 2013a; Metsämuuronen & Tuohilampi, 2014; Ukkola & Metsämuuronen, 2019; Ukkola et al., 2020). This is caused by the fact that the preliminary concepts related to mathematics are learnt at home or during the preprimary education, and these conditions may vary dramatically (see Ukkola et al., 2020). Hence, some children enter the school with no or very limited knowledge of basic mathematical concepts while some may be already at the level of grade 3 (Ukkola & Metsämuuronen, 2019; 2021). The reasons for this deviance are discussed here, focusing on the factors related to the child and the home background.

### 2.1 Factors explaining the preconditions of mathematics in literature

Several individual factors have been shown to affect the development in general and in mathematics in specific. Some of these are sex (see, e.g., Metsämuuronen 2017a; Niemi et al., 2020, 2021), and language background including medium of instruction being the mother tongue (first language, L1) or the second language (L2) (see, e.g., Kuukka and Metsämuuronen, 2016). Other important factors found to explain the competence are relative age of starting the school (see, e.g., Dhuey et al., 2019; Kivinen, 2018; Ukkola et al. 2020), attitudes toward school and self-efficacy (see, e.g.,

Aunola, Leskinen, & Nurmi, 2006; Bandura, 2012; Lerkkanen et al., 2010, 2012; Tuohilampi and Hannula, 2013; see in-depth in Ukkola et al., 2020).

Three factors related to child's home background are found to be important in explaining the pupil's school performance: education of the parents (see, e.g., Kivinen, Hedman, & Kaipainen, 2012; OECD 2015), economic factors (see, e.g., Erola, Jalonen, & Lehti, 2016; Paju, 2020; Palomäki et al., 2016; Sirniö, 2016), and genetics including inheritance related to mathematics (see, e.g., Dilnot et al., 2016; Malanchini et al., 2020). The first two are commonly combined as factors related to socioeconomic status (SES), and the latter has been important factors in explaining learning disabilities, for example (see, e.g., Eklund, 2017).

All in all, many factors related to a child—of which many are given, and of which the child cannot affect at all—are related to the early childhood development in general and mathematics development in specific. In-depth discussion of all these matters is found in Ukkola and colleagues (2020). These are discussed further in the empirical section.

## 2.1 What is known of the combined factors explaining the low level of preconditions of mathematics?

Ukkola and colleagues (2020) collected quite a variety of possible variables explaining the high and low levels of preconditions in general at grade 1. They sought a simple model, a kind of check list type of presentation of the factors predicting the exceptionally low level of preconditions in the population. Based on logistic regression analysis (LRA) and decision tree analysis (DTA), they came up with five binary variables explaining the low performance in the test of preconditions in mathematics and language combined (Table 1).

The strongest predictor for the low performance in the test of preconditions is whether the child was decided to be on intensive or special support even before the school age. The risk of these children to belong to the lowest quartile (Q1) is 4.6 times higher and to the lowest decile (D1) 5.3 times higher than when it is not the case. Second strongest predictor is the L2 status with 3.3- and 4.2-times risk, respectively. Other factors such as learning disabilities of the parents, relatively young school starting age, and guardians' low education give 1.5- to 2.0-times risks for a child to belong to the group of exceptionally low preconditions in general. Notably, the explaining powers of the models are not very high ( $R^2_{Adj} = 0.12 - 0.13$ ) referring to the fact that even though the tendency is clear, nothing is determined even if the child



happens to be born with less advantageous genes and a “wrong” time of the year or in a “wrong” country.

**Table 1.** Five main factors explaining the low level of preconditions in mathematics and language combined at grade 1 (Ukkola et al., 2020)

Variables in the model <sup>1</sup>	B <sup>2</sup>	risk to be at Q1 (lowest quartile)	risk to be at D1 (lowest decile)
Constant	547		
Support in three levels (1 = decision on intensive or special support, 0 = general support meant for all pupils)	-65	4.58	5.30
Status for Finnish/Swedish as a second language (L2 status) (1 = registered L2 status, 0 = no L2 status)	-63	3.29	4.17
Learning disabilities in the close family (1 = at least one type of learning disability in parents, 0 = no learning disabilities in the close family)	-36	1.99	1.83
Relative age of starting school (1 = months 9–12, 0 = other months)	-35	1.76	2.01
Education of the guardians (1 = both or either of the guardians have basic education or vocational qualification, 0 = other alternatives)	-30	1.71	1.50
predicted level if in group 1 in every factor	320		
Explaining power $R_{Adj}^2$	0.12	0.13	0.13

- 1) Variables are ordered by the risk related to Q1
- 2) regresson weight

### 3 Quest for common standards for mathematics

Assessing the *absolute* level of mathematical skills or thinking is not as simple as sometimes it is thought to be. Specifically, the task is even more difficult when test takers are young and there is no obvious measurement stick which would tell what “good” or “high level” is. The most obvious challenge is that there is no commonly accepted general framework for proficiency in mathematics.

Metsämuuronen (2018) suggested a common framework for mathematics (CFM) based on the levels used in the common European framework of reference for languages (CEF or CEFR; <https://www.coe.int/en/web/common-european-framework-reference-languages>)—after all, mathematics is a kind of universal language, and mathematical skills tend to cumulate. In CFM, the domains are reduced to three elements: 1) proficiency in mathematical concepts (M1), 2) proficiency in mathematical operations (M2), and 3) proficiency in mathematical abstractions and thinking (M3). The rationale for the first two criteria is obvious: to master even the

simplest and most mechanical mathematical operation, a certain level of proficiency in mathematical concepts is needed: the concepts of numbers and their representations, consecutive nature of the numbers, and certain basic shapes such as the triangle, square, and circle. The rationale for proficiency in mathematical abstractions and thinking is that the essence in mathematical proficiency (maybe except at a theoretical level) is to transform everyday life challenges into a mathematical form and solve the problems by using mathematical operations. Without proficiency in mathematical abstractions and thinking, the proficiencies in concepts and operations are largely useless; one may know how to do a mathematical operation (such as derivation) but have no idea when or why to use it.

The basic mathematical concepts, operations, as well as the elements of mathematical abstractions and thinking are usually hierarchically organized in the normal educational process. For example, to manage powers, the procedure of multiplication is needed and to master multiplication, the procedure of addition is needed. Hence, we understand that it is wise to start teaching and learning mathematics with concrete things such as addition and subtraction of the natural numbers before introducing decimals and rational numbers.

The standard levels in CFM are based on this logic which are divided into levels A, B, and C (Table 2). The level A refers to the elementary and basic level with the relevance to the *everyday life*, B refers to an advance level with relevance to the *further studies* in several professional areas like statistics, engineering, or economics, and C is the *professional* level mathematics needed either in practical fields (like that of statisticians, advanced researchers, economists, or engineers) or in the theoretically oriented fields (like that of professors or researchers of pure mathematics, physics, astronomy, or chemistry).

As far as this article is concerned, only level A1 is relevant at the beginning of the school even though there may be some prodigies among the pupils. The descriptions and stages in CFM are based on the national core curricula of mathematics in Finland (EDUFI, 2004, 2014 for the basic education; EDUFI, 2003, 2015 for the upper secondary general education).

**Table 2.** Brief descriptions of the CFM levels (Metsämuuronen, 2018)

CFM level (main stages)	CFM level	Short Description
A1 Elementary proficiency	A1.1	First stage of elementary proficiency
	A1.2	Developing elementary proficiency
	A1.3	Functional elementary proficiency
A2 Basic proficiency	A2.1	Developing of basic proficiency
	A2.2	Functional basic proficiency
B1 Advanced proficiency	B1.1	First stage of advanced proficiency
	B1.2	Developing advanced proficiency
B2 Functional advanced proficiency	B2.1	First stage of Functional advanced proficiency
	B2.2	Functional advanced proficiency
C Professional level	C1	Basic Professional level
	C2	Advanced Professional level

In CFM, the first measurable level is A1.1 (First stage of elementary proficiency) where the basic elements needed in mathematics such as the numbers and basic shapes related to geometry are identified. A1.1 refers to the level at which the rudimentary basic elements of mathematical proficiency are mastered. At this level, among others, one is familiar with the numbers, but the use in mathematical operations is very limited; one recognizes the basic two-dimensional shapes (circle, square, triangle) and their three-dimensional counterparts (ball, box, and pyramid) and can couple their name with pictures; one can express some limited mathematical expressions, such as the order of numbers; one knows the importance of numbers in stating amount and order; one knows how to write numbers but the proficiency in using formulated mathematic expressions is very limited.

The empirical section is specifically about pupils *below* level A1.1. These pupils have the most disadvantageous start for their mathematical career although they may also give the most joy to the teacher when noticing how well they advance in school despite the low level at the beginning.

If someone is at a level lower than A1.1, from the mathematical *concepts* viewpoint, he or she may not (adequately) know the numbers in the range 1–10; may not be aware of the consecutive nature of numbers; may not be able to name the basic forms of the circle, square, triangle, ball, box, and pyramid; and may have no understanding of the basic concepts of length, mass, volume, and time.

From the mathematical *operations* viewpoint, a person below level A1.1 may not be able to recognize or write the numbers; may not understand the consecutive order of numbers; may not be able to categorize the basic shapes into groups without messing with different sizes, colors, and positions; may not be able to couple the

names of basic shapes with pictures; and may not know how to measure length, mass, and time in the everyday life.

From the mathematical *abstractions and thinking* viewpoint, a person below level A1.1 may not have the basic understanding of the concepts of adding, subtracting, dividing, or multiplying; may not have the basic understanding of unseen numbers (for example, what number is missing in the consecutive order); may not have the basic understanding how to place things in order, to find opposites for things, to classify things according to different attributes, or to state the location of object for example by using the words *above, below, on the right, on the left, behind, and between*.

Obviously, only a new-born baby may be at the stage where the mathematical thinking or understanding of concepts and operations would be non-existent—all children starting the school have *some* mathematical preconditions and skills as discussed above.

## 4 Methodology

The general methodological issues related to the dataset used in the empirical section are discussed in detail by Metsämuuronen and Ukkola (2019). Some points relevant to this article are highlighted in Section 4.1 about the sample and datasets while Section 4.2. is about the test of mathematic. Section 4.3 describes the procedure of standard setting and Sections 4.4 and 4.5 describe the practicalities related to the analysis itself.

### 4.1 Sampling and data

A dataset of  $n = 7770$  pupils from grade 1 was collected in August 2018 from 264 schools selected by using stratified random sampling. The selected schools comprise 13% of all schools teaching grade 1 and the pupils are 19% of all grade 1 pupils in Finland. Swedish population was oversampled (28% of the Swedish-speaking schools) for a relevant analysis of this minority. Of the pupils in the target group, 97.5% participated in the test.

Part of the information concerning the child was provided by the guardians of the child. This dataset comprises  $n = 4,316$  children (56% of the pupils) and it includes information of the parents as well as such information of the child that was difficult to extract from the child, for example, concerning their interests. Hence, some

analyses can be done by using the whole dataset while some other interesting variables are restricted to a smaller number of pupils and, in the latter case, the dataset is slightly biased toward higher-educated families; as usual, guardians with a higher educational level seemed to have been more active in answering the questionnaire (see closer Metsämuuronen & Ukkola, 2019). Relevant characteristics of the dataset are collected in Table 3.

Using pupils' ID numbers, relevant information was added to the dataset from the national KOSKI-database. This included information such as home language, L2 status, and information concerning the 3-stage support.

#### 4.2 Test items, validity, and reliability of the test score

The content of the mathematics test was based on content areas in the National core curricula for preprimary education (EDUFI, 2016) and for basic education (EDUFI, 2014). Based on these norms, the contents of the mathematic test comprised of three main areas: geometry and measurement, numbers and calculation, and mathematical thinking (Table 4). From the construct validity viewpoint, the test comprises all areas of the “theoretical framework” from the core curricula.

The sub-test of mathematics comprises 58 items totaling 62 points. The lower bound of reliability of the test was  $\alpha_R = 0.88$  by coefficient alpha and, after correction for deflation by using Somers'  $D$  instead of Pearson correlation in the coefficient (see Metsämuuronen, 2020, 2021, 2022a, 2022b; Metsämuuronen & Ukkola, 2019),  $\alpha_D = 0.94$ . Hence, in general, the score is accurate enough to discriminate the test takers from each other.

After a pre-trial, two task types were selected to the final test: “press” and “move” (see Figures 1 and 2). The pupils did the assessment tasks in the school's language of instruction using a tablet or a computer. The tasks were speech-instructed. Each pupil logged into the testing system through a unique sequence of graphical symbols and selected an avatar (such as a robot) to lead into the test (and “speaking” the instructions). Children learnt quickly how to use these two task types by a training sequence before the test. Teachers were instructed to help the child if some technical challenges occurred but not to interfere with the answering process. After selecting the item, an arrow appeared automatically. The child pressed the arrow to move to the next task.

**Table 3.** Characteristics of the datasets of grade 1

Variable		Whole dataset ( <i>n</i> )	Pupils <i>n</i> = 7770 (%)	Guardians <i>n</i> = 4316 (%)
Sex	Girl	3875	49.9	49.9
	Boy	3895	50.1	50.1
Instruction language	Finnish	6902	88.8	91.1
	Swedish	868	11.2	8.9
Syllabus	Finnish	6405	82.4	
	Swedish	834	10.7	
	Fin/Swe as second language (L2)	531	6.8	
Regional state administrative agency	South Finland	3015	38.8	38.2
	South-West Finland	917	11.8	11.8
	East Finland	732	9.4	8.7
	West and Middle Finland	1672	21.5	22.2
	North Finland	780	10	10.3
Type of municipality	City	5468	70.4	70.5
	Population density area	1184	15.2	15.7
	Rural	1118	14.4	13.8
L2 status	No	7239	93.2	95.1
	Yes	531	6.8	4.9
Three-stage support	General support	6971	89.7	92.2
	Intensive support	521	6.7	5.7
	Specific support	278	3.6	2.1
Learning disabilities in parents	No learning disabilities	3125		72.4
	One type of learning disability	720		16.7
	Several types of disabilities	471		10.9
Highest education in the family	Basic education	48		1.1
	Vocational education	927		21.5
	Matriculation examination	343		8.0
	Polytechnic education	1346		31.2
	University education	1535		35.6
	Else	111		2.6
Level of preconditions	Score in mathematic test	500		514

**Table 4.** Contents of the mathematics test

Domain	Topic	Number of items	Reliability ( $\alpha_R$ )	Reliability ( $\alpha_D$ )
Mathematics as whole		58	0.879	0.940
Geometry and measurement	Geometry	9		
	Measurement	10		
	Time and Clock	4		
Numbers and calculation	Calculation	17		
	Numbers	8		
Mathematical thinking	Oral tasks	17		
	Reasoning	8		
	Relations	17		

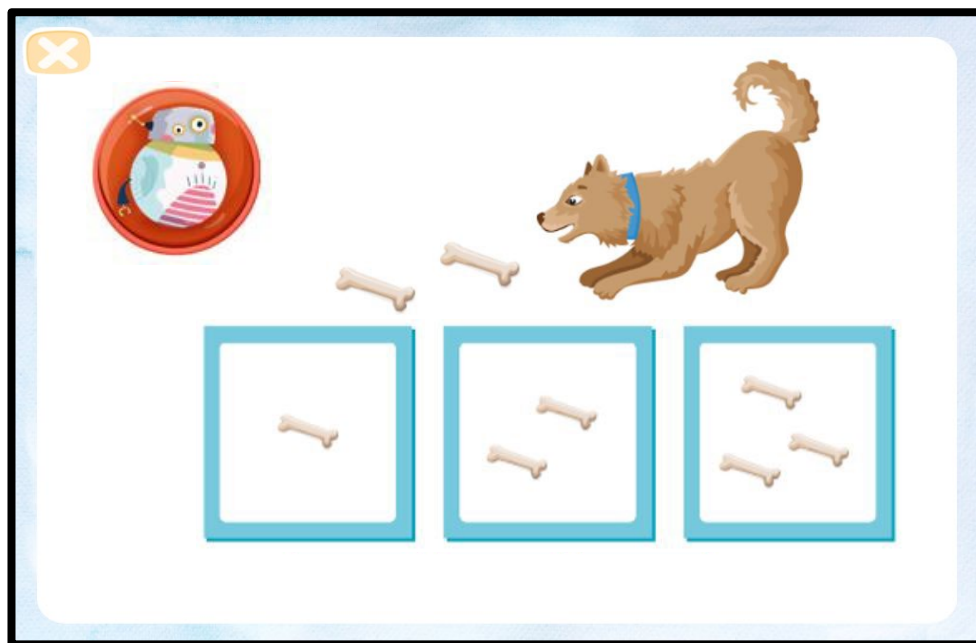


Figure 1. A “Press” type of task (“Press the figure which has one bone less than the dog has”)



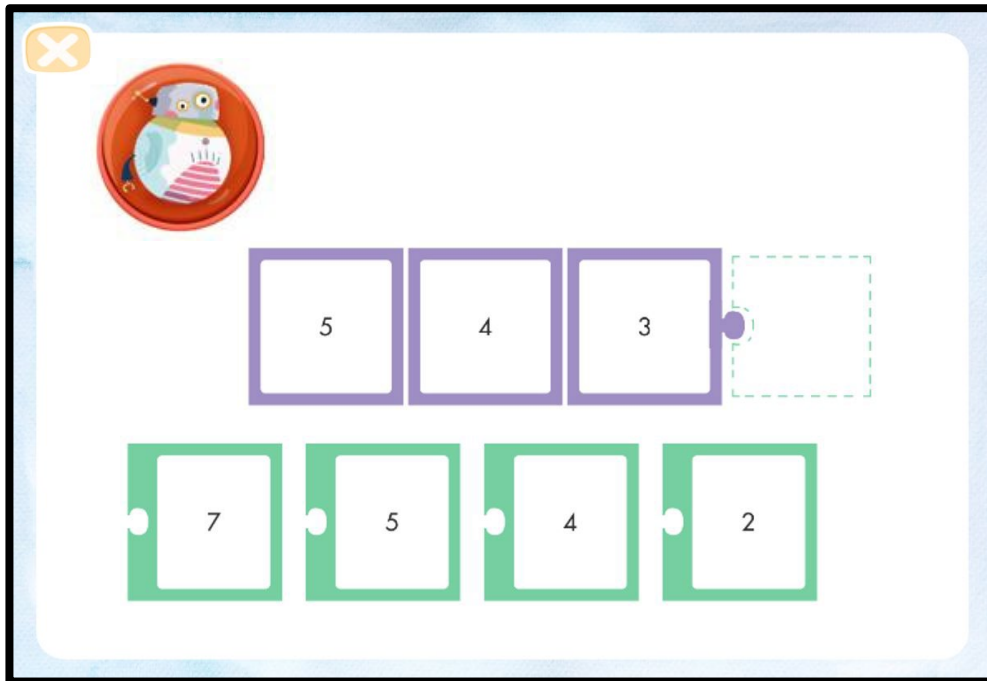


Figure 2. A “Move” type of task (“Move the correct number to the queue”)

### 4.3 Standard setting

The original analysis (see Ukkola & Metsämuuronen, 2019; Ukkola et al. 2020) was based on norm-reference assessment. For the re-analysis, a standard setting was administered by using a method called 3TTW (Three-phased Theory-based and Test-centered method for the Wide range of proficiency levels, Metsämuuronen, 2013b).

At the first phase of 3TTW, items were classified on different bins of standard systemic based on the (theoretical) content of the item; first based on three criteria (concepts, operations, and thinking) and second, based on the standard level (A1.1, A1.2, and A1.3). Only in the criteria on mathematical abstraction and thinking, it was possible to find items that fit the level A1.3. These items were more demanding where a semi-complicated real-world problem was needed to be transformed into a mathematical form and to solve for instance in “*In the morning, the thermometer showed +2 Celsius. During the school day, it dropped six degrees. What is the temperature after the school day? Press the correct number.*” For a possible interest of a reader, in this type of item, 12.5% of the pupils of the grader 1 were able to give a correct answer at the beginning of school.

At the second phase, items belonging to a same bin were summed up. The sums were transformed into a form that indicated whether the test taker had reached the level of proficiency required for a specific level of standard in a specific criterion. For

this, a proper cut-off in the sums was set to mark the needed level of proficiency for the standard level. Usually, these boundaries are named as “weak pass”, “pass”, and “strong pass” (e.g., Van der Schoot, 2009)—strong pass may mean that one needs to score at least 80% of the total score in the bin. A minimum boundary was set to 50% correct, that is, to “weak pass”: to belong to a certain standard level, the test taker needed to solve at least 50% of the tasks correctly (see Figure 3).

At the third phase, each test taker got his/her profile of passing and failing in the levels of standard systemic. In most cases, the profile was “pure” in a sense that if one was able to solve more-demanding tasks, also the less-demanding tasks were solved. Then, it is straightforward to conclude that if a test taker can show proficiency enough for levels A1.1 and A1.2 but not for A1.3, a credible proficiency level for the test taker is A1.2 (see detailed, Metsämuuronen, 2013b).

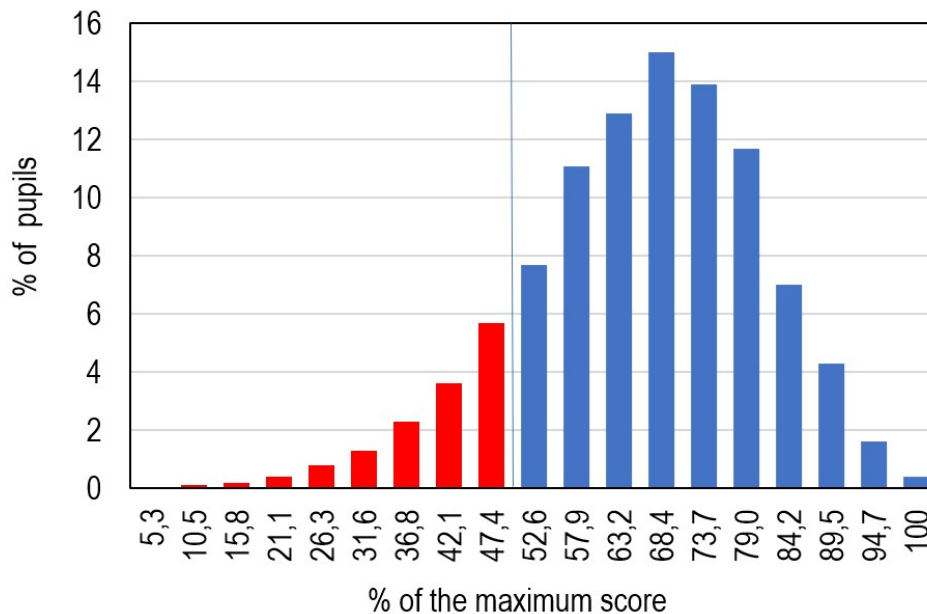


Figure 3. Distribution of proficiency in M3, level A1.1 items summed, and the boundary of a weak pass (50% correct)

#### 4.4 Variables used in the analysis

The score of mathematics is formed of the raw score by one-parameter item response theory (IRT) modelling, that is, the Rasch modelling. The outcome (theta score) is a logistic transformation of the raw score. The original theta score is a standardized normal variable where the average scorer gets the value 0. This score is further transformed to a T10 form, that is,  $Y = 100 \times X + 500$  leading to a score where the average test taker gets the score 500 and the standard deviation is 100. The same

transformation and mechanics are used, for instance, in PISA- and TIMSS inquiries (see, e.g., PISA, 2019). In what follows, also some other test scores such as different sub-tests related to the medium of instruction of the school and attitude scales are used in analysis. Validity and reliability of these tests are described in Metsämuuronen and Ukkola (2019).

For a re-analysis of the dataset, a standard setting was administered (see above), and those pupils were detected who showed the lowest absolute level of preconditions of mathematics at grade 1 in all criteria ( $n = 608$ ; 7.8% of the pupils). In theory, these pupils are below the lowest measurable level of proficiency in mathematics at the beginning of grade 1. Naturally, they have some proficiency in mathematics—in some cases maybe almost half of the task solved—but not enough to reach the lowest standard level A1.1 in any of the areas on the criterion systemic. This dummy variable (later “below A1.1”) is mainly discussed in what follows.

The analysis is mainly exploratory in nature. Hence, relevant descriptive variables such as sex, relative age of school start, and family factors are used to profile these pupils with the least advantageous start of the mathematical studies in school. Finally, by combining the statistically significant predictors, a model parallel to Table 1 is formed to predict the grouping of the lowest level preconditions in an absolute sense, that is, the group “below A1.1”.

#### 4.5 Methods of analysis

Three main analytical tools are used in the analysis: a data mining tool decision tree analysis (DTA), traditional logistic regression analysis (LRA), and traditional general linear modeling (GLM) in IBM SPSS environment. These methods are generally known and, hence, there is no need to describe them further (see, e.g., Metsämuuronen, 2017b). In LRA, standard statistical procedure with conditional selection of variables is used with Nagelkerke’s (1991) adjustment for the explaining power ( $R_{Adj}^2$ ). GLM is used mainly as one-way ANOVA; in post hoc analysis, Šidák’s (1967) procedure is used; in the case, it gives more plausible correction for  $p$ -values than the traditional Bonferroni correction (see discussion in Metsämuuronen, 2017b). For effect sizes, Cohen’s  $f$  (Cohen, 1988) is used; the classic, rough boundaries for small-, medium-, and large effect size are  $f < 0.1$ ,  $f = 0.2–0.3$ , and  $f > 0.4$ , respectively. In DTA, CHAID algorithm (Kass, 1980) is used, and child nodes with three levels were allowed as is default in SPSS (see detailed, Metsämuuronen, 2017c).

## 5 Results

### 5.1 Overall distribution of preconditions in mathematics at the beginning of the first grade

Overall, the distribution of test score of preconditions in mathematics forms a slightly widened normal distribution (Figure 4) referring to the fact that whole population may be comprised of three to four different normal populations with slightly different means. This seems to fit what was previously noted by Metsämuuronen (2017a; see also Metsämuuronen & Tuohilampi, 2014): children starting the school in Finland seem to form four populations. *Developing pupils* have no or very thin idea of mathematical concepts or thinking—this group is very small in Finland. *Beginner pupils* have some academic preconditions and understanding of mathematical concepts and thinking although those may be very limited—this group is also rather small in Finland. The target group in this article consists mainly of pupils in these two groups. *Normally developed pupils* form the main population. They recognize or master basic concepts such as natural numbers in a limited range and can name basic forms such as triangle, circle, and square; they may be able to solve simple mathematical problems by using adequate mathematical operations; and they may have basic understanding of measuring mass and time, for instance. *Advanced pupils* form the highest performing segment of the cohort. Their mathematical performance at the beginning of school may already be partly at the level of grade 3. Some pupils in this group may be categorized as *exceptionally advanced pupils*. Notably, in the dataset the ultimately highest and lowest-performing pupils were boys. Also, in both extremes (scores  $< 200$  and  $> 800$ ), the number of boys is twice that of girls. This fits with the greater male variability hypothesis discussed by, e.g., Baye and Monseur (2016), Johnson, Carothers, and Deary (2008), Machin and Pekkarinen (2008), and O’Dea and colleagues (2018). However, the number of ultimately performing pupils is rather small.

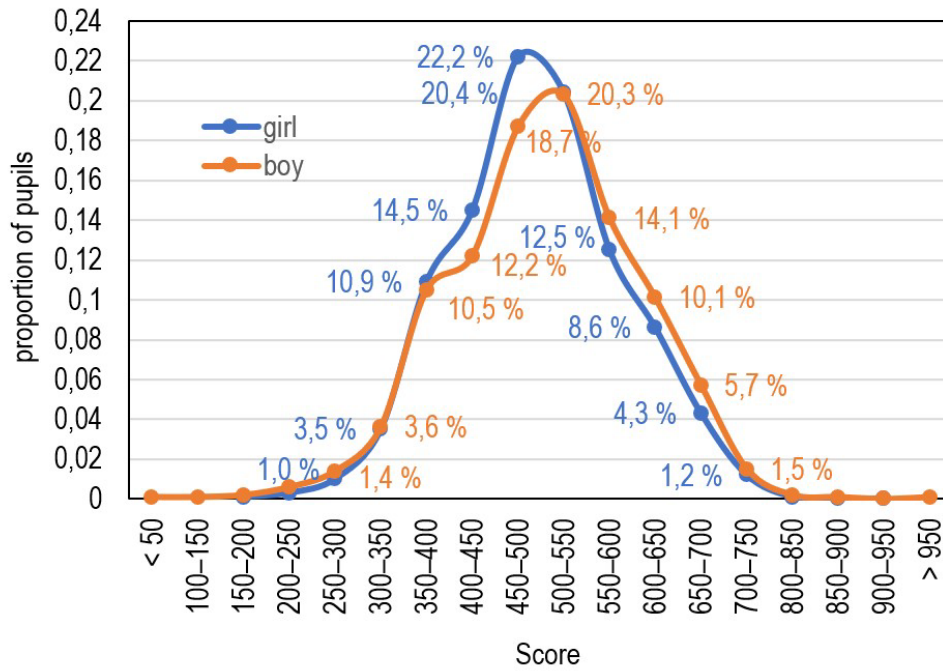


Figure 4. Distribution of the score of preconditions in mathematics at grade 1

After the standard setting, the distributions of proficiency levels are as in [Figure 5](#). In criteria M1 and M3, 12–13% of the pupils fall in the category “below A1.1” while, in criterion M2, 51% of pupils fail to reach the level A1.1. This refers to the fact that, at the beginning of grade 1, pupils may know well the basic natural numbers and recognize and name basic shapes, and they may show some elementary mathematical thinking, but they cannot use much mathematical operations. This makes sense because the mathematical operations are not taught in the preprimary education in Finland; these are taught in school.

By labeling the standard levels with ordinal 0, 1, and 2 in M1 and M2 and 0, 1, 2, and 3 in M3 and summing up the levels of different criteria, we get a rough distribution of absolute proficiency levels for each pupil ([Figure 6](#)). While 7.8% of the pupils fall into the category “below A1.1”, 22.5% of the pupils appeared to be at the levels A1.2 or A1.3 in all criteria. Notably, the proportion of pupils in the highest category is exceptionally high because the category consists, factually, of *two* categories, “A1.2 in all criteria” and “A1.2 in criteria M1 and M2 and A1.3 in criterion M3”. The middle levels are formed by varied combinations of the standard levels and, hence, their interpretation is ambiguous.

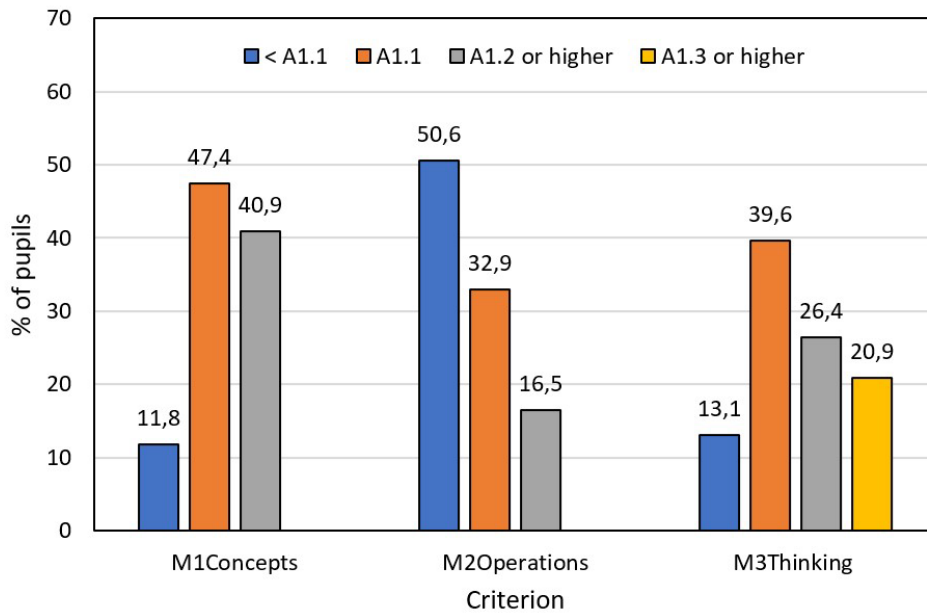


Figure 5. Distributions of standard levels of different criteria in mathematics at grade 1

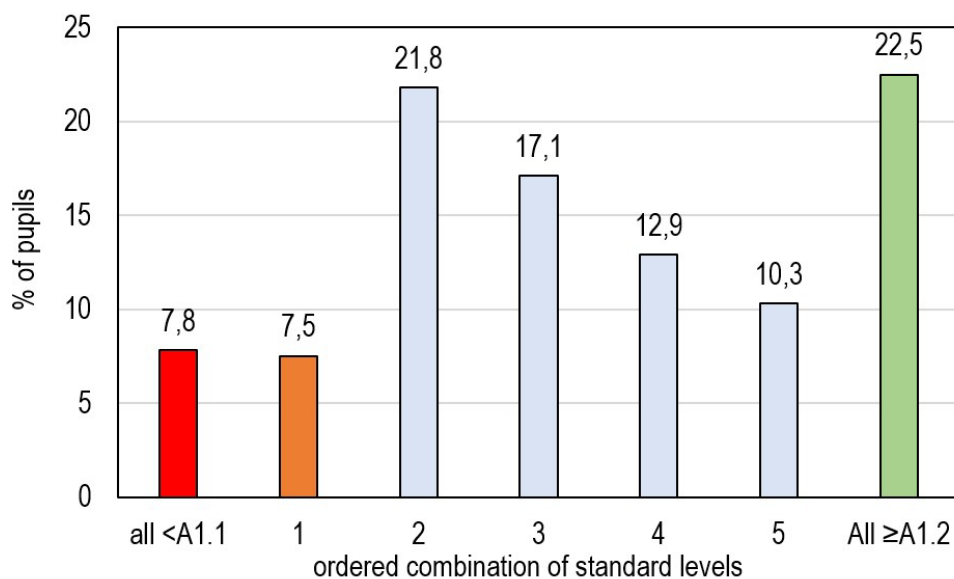


Figure 6. Distributions of combined standard levels in mathematics at grade 1

The average score in the group “below A1.1” was 329 (std. dev. 54.57) and in the highest achieving group “A1.2 or more in all criteria”, it was 628 (std. dev. 58.50) (Figure 7). The difference between the groups is, obviously, statistical significant ( $F(6, 7763) = 5422.54, p < 0.001$ ) and the difference between the extreme groups is remarkable ( $f = 2.04$ ). Also, the average scores in each level of ordered proficiency levels differ from each other statistically significantly (GLM, post hoc tests, all  $p < 0.001$ ).



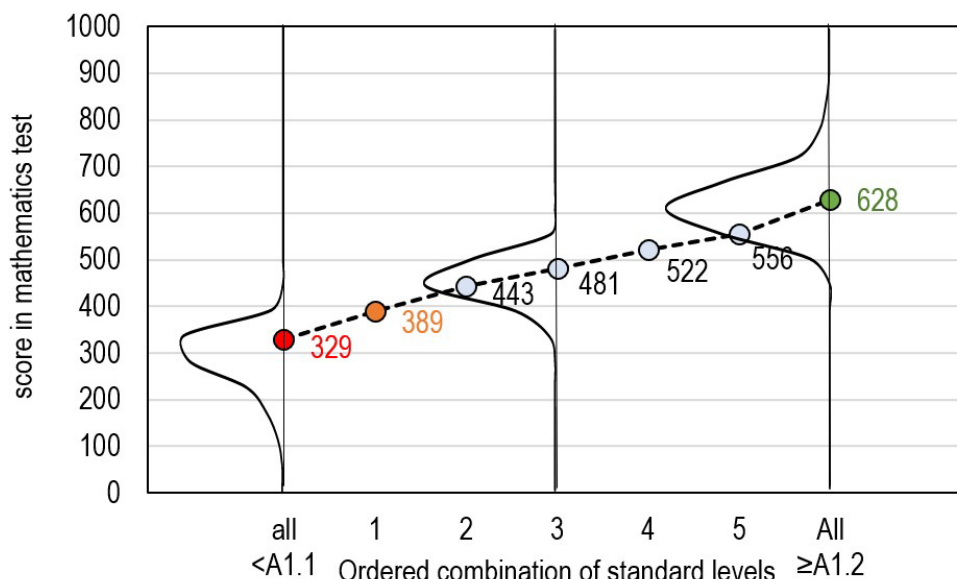


Figure 7. Relation of the test score and the combined standard levels in mathematics at grade 1 with observed distributions for selected levels

## 5.2 Personal factors characterizing the pupils with very low preconditions in mathematics at the beginning of the first grade

When focusing on the pupils below the lowest measurable standard level ( $n = 608$ ), DTA suggests that these pupils scored low also in the general test that combined mathematics and the medium of instruction of the school (Finnish/Swedish). Hence, these pupils low-performed not only in mathematics, but they were at a lower level in preconditions for the school in general. Because the test score in the medium of instruction of the school correlates almost one to one with the total score ( $r = 0.996$ ) caused by the fact that almost all items include a linguistic component (see Metsämuuronen & Ukkola, 2019; Ukkola et al., 2019), it is expected that deficiencies in proficiency in the medium of instruction may explain well the low performance in mathematics.

Of all 17 sub-tests related to proficiency in language (see Metsämuuronen & Ukkola, 2019), DTA suggests that the low score of proficiency in *listening comprehension* predicts inclusion in the low-achieving group the strongest; 68% ( $n = 415$ ) of the pupils in the group “below A1.1” came from the group where the score of the listening comprehension was below 372.19. By LRA, the main effect of the listening comprehension (dummied into lower and higher than score 372,19) is remarkable: the risk of belonging to the group “below A1.1” is 44,2 times higher if the pupil scored 372 or lower, and the explaining power of the simple model is high



( $R_{Adj}^2 = 0.408$ ). Also, in GLM, the effect size is high ( $f = 0.71$ ). Notably, 72.6% of the pupils in this group were *not* immigrants with an L2 status. Logically, if the language skills are underdeveloped in general, the pupils are not expected to master the more demanding specific mathematical vocabulary either. Hence, they get a low score in mathematics too.

L2 status also has its own—although small—main effect in predicting belonging to the group “below A1.1”. When L2 status is added to the model of LRA with the dummied score of listening comprehension, it still appeared to be a statistically significant independent predictor ( $p = 0.003$ ), with the risk index 1.5. Then, it has an effect, but the effect is small in comparison with the low language comprehension.

One obvious possible factor explaining the low absolute achievement level is the decision regarding *intensified or special support*. In Finland, this decision could be made during preprimary education based on the obvious signs for slow learning. Intensified and special support are given already before school and, in many cases, the need is still there when the school starts. In the dataset, the level of 3-stage support appears to explain significantly (GLM,  $F(2, 7767) = 219.14$ ,  $p < 0.001$ ) and remarkably ( $f = 0.24$ ) belonging to the group “below A1.1”: in the group with a need for intensified support, 20% belonged to the group “below A1.1” and, to the group with a need for special support, 35%. For further analysis, these two groups are combined (Figure 8).

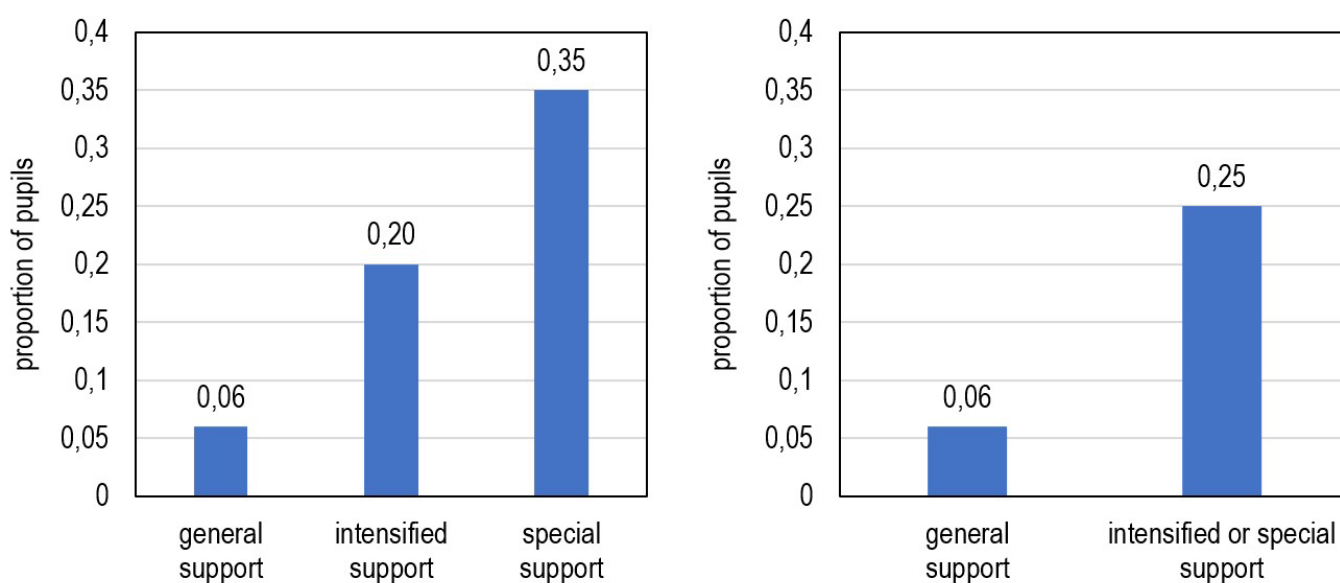


Figure 8. Proportions of pupils belonging to the group “below A1.1” at the levels of the 3-stage support

Attitudes and emotions are shown to be related to performance although the direction of the effect is not easy to determine unambiguously. Namely, we do not know whether high proficiency level creates a positive attitude, is it the other way around, or is it reciprocal. For grade 1 pupils, the attitude items related to mathematics were notably simple: “*Counting is...*” and “*I can count*” and “*The tasks were easy*” with smile faces with 5-point Likert type of scale anchored to 1–5 (see Metsämuuronen & Ukkola, 2019). Still, “counting” may be rather concrete for many young children: they might think, for example, listing numbers in numerical order.

Of the dimensions of attitude (the whole test, general attitude toward math and language, attitude toward mathematics, attitude toward language, and self-efficacy), the *attitude toward mathematics* appeared to be the factor explaining the best belonging to the group “below A1.1” in DTA. The lower is the mean in the attitude scale, the higher the probability to be found in the group “below A1.1”. Division of the attitude scale into four groups explains the belonging statistically significantly (GLM,  $F(3, 7457) = 53,62, p < 0.001$ ) although not remarkably ( $f = 0.15$ ). Again, we do not know how much the result is related to a factual realistic understanding of the child: “I really was not able to solve the tasks, hence, the tasks were not easy to me”. If so, it shows that the children even at the grade 1 seem to be able to do realistic evaluation of their capabilities. For the later use, the attitude variable is dichotomized from lower and higher than the score 3.333 as suggested by DTA.

Of other personal factors discussed in the introduction, neither *sex*, more *relative school starting age* nor any of the *hobbies*—not even programming or reading in home (see the variables in Metsämuuronen & Ukkola, 2019)—did explain the belonging to the group “below A1.1” in this dataset. This may be partly explained by the fact that the information concerning the latter variables were provided by the guardians, and this reduced the number of pupils belonging to the group “below A1.1” from  $n = 608$  to  $n = 246$  pupils.

### 5.3 Family factors characterizing the pupils with very low preconditions in mathematics at the beginning of the first grade

Two relevant sets of variables related to parents and guardians in explaining the pupils to belong to the lowest achieving group are discussed here: guardians’ educations on the one hand and potentially inherited disabilities from the parents on the other. This information is obtained from the guardians’ questionnaire and reduces the number of pupils to almost a half and in group “below A1.1” from  $n = 608$  to  $n =$

246.

The educational background of two guardians (“mother or other guardian” and “father or other guardian”) was asked being at one of the five levels: basic education, vocational education and training, matriculation examination, polytechnic, and university degree. Few guardians also selected the alternative “other”—this alternative seems to be a more typical selection by guardians from immigrated families. From this information, several combinations of educational background were derived to explain the general level of preconditions in mathematics (see Ukkola et al., 2020). Here, the original variables are used.

*Mother’s education* appeared to be a better predictor ( $\chi^2(2) = 41.82, p < 0.001$ ) than that of father’s ( $\chi^2(2) = 29.37, p < 0.001$ ). Although DTA suggests three groups for mother education (basic, secondary, and tertiary education), after the correction in  $p$ -values by using Šidák’s procedure, GLM suggests that only the group of mothers with just *basic education* (or “missing”) differs from the other groups (post hoc,  $p = < 0.001$ ; for the other groups,  $p > 0.05$ ). Of the children with this background, 16.8% belonged to the group “below A1.1” while in two other groups the percentage is 5–6%. Effect size is small though ( $f = 0.095$ ).

Another interesting family-related factor explaining the belonging to the lowest-achieving group is the possibly inherited learning disabilities. Five different types of disabilities were given as alternatives in the guardians’ questionnaire: linguistic (such as dyslexia), mathematical (such as dyscalculia), concentration, perception, and social challenges. Of these, linguistic and mathematics disabilities did not explain the lowest performance although they may, in general, influence performance. However, pupils belonging to the group “below A1.1” were slightly more likely to have *parents with concentration problems* (23.6% vs. 11.4%,  $\chi^2(1) = 31.99, p < 0.001$ ). In LRA, it shown 2.4 times risk although with low explaining power ( $R_{Adj}^2 = 0.017$ ). In GLM ( $F(1, 4314) = 32.21, p < 0.001$ ), the effect size is small ( $f = 0.08$ ).

## 5.4 Outline of the results

By combining the results from Sections 5.1–5.3, we may conclude that, of the variables used in the analysis, *deficiencies in language*—specifically a low level of understanding of spoken language, also indicating a lack of adequate vocabulary related to mathematics—is the most powerful factor explaining why the precondition level on mathematics in pupils remained lower than the lowest measurable level (below A1.1 in all criteria). This variable alone explains 41% of the variance in the

dataset. In what follows, all variables from the previous sections showing statistically significant prediction power are collected and dichotomized to make a simpler model in LRA. The variables in the modelling included the *score below 372 in the test of listening comprehension*, *L2 status*, *decision on intensified or special support*, *attitude toward mathematics*, *mother's education in three categories*, and *concentration problems of parents*.

During the modeling, mother's education did not have a major effect and it was dropped in the statistical process. Also, using learning disabilities reduces the dataset to almost half which reduced the explaining power of the models. Hence, in the final model only four variables found from pupils' dataset were kept. The outcome is summarized in [Table 5](#).

**Table 5.** Four factors explaining the low level of preconditions in mathematics

Variables in the model <sup>1</sup>	B	Significance	Risk to belong to the group < A1.1 EXP(B)
Constant	-3.877		
Score in the test of Listening comprehension (1 = score ≤ 372.19. 0 = score > 372,19)	3.351	< 0.001	28.53
3-stage support (1 = decision on intensive or special support, 0 = general support meant for all pupils)	0.777	< 0.001	2.176
L2 status (1 = registered L2 status, 0 = no L2 status)	0.461	0.003	1.586
Attitude toward mathematics (1 = mean score < 3.333, 0 = mean score > 3.333 in the scale of 1–5)	0.457	< 0.001	1.579
Explaining power $R_{Adj}^2$	0.40		

1) Variables are ordered by the risk

After knowing the score of the listening comprehension, the 3-stage support still gives 2.2 times risk, while L2 status and a low score in a simple test of attitudes toward mathematics have 1.6 times risk to belong to the group “below A1.1” in both mathematical concepts and procedures as well as in abstractions and thinking. The explaining power of the model is reasonably high ( $R_{Adj}^2 = 0.40$ ). Notably, low score in the test of listening comprehension, alone, had even higher explaining power ( $R_{Adj}^2 = 0.41$ ) with a higher risk value of 44.2. The reason for the non-intuitional higher explaining power by a smaller model is that other variables include missing values causing reduction in pupils included in the analysis.

## 6 Discussion

An obvious conclusion of the analysis is that not all variables explaining the low level of general preconditions in mathematics and language (Ukkola et al., 2020; see Table 1) are valid in explaining the *absolute* low performance in mathematics. A somewhat surprising although understandable finding is that low proficiency in listening comprehension appears to be a key factor in explaining the low proficiency in mathematics too. It is understandable that if the language skills are underdeveloped in general, the pupil is not expected to master the rarer, specific mathematical vocabulary either, hence the low score in mathematics too. These two phenomena, proficiency in mathematics and proficiency in listening comprehension are not totally independent in the dataset though; part of the items in the mathematics test were used also as part of listening comprehension; after all, nearly all mathematics items also included a component of understanding concepts of mathematics and all instructions were given orally. Hence, further studies of independent tests of mathematics and language would be beneficial. Anyhow, we may predict that all activities increasing the language skills, specifically, of the wider vocabulary in the early childhood may also increase mathematical comprehension. The specific vocabulary related to mathematics may need some conscious concentration from guardians and preprimary teachers.

An obvious limitation of the study is that relevant pieces of information concerning the child was collected from guardians and this information was given only for around half of the pupils and of these, more likely, for better performing children. Hence, with relevant variables explaining pupils belonging to the group “below A1.1”, the number of pupils was reduced from  $n = 608$  to  $n = 246$ . In future phases of the longitudinal setting, it is aimed to collect more information from those families that did not answer the questionnaire in the first phase. Hence, the results reported in this article may get more power, specifically, when it comes to parents’ and guardians’ role in the early development of the child.

Teachers in the primary education are facing an interesting challenge at the beginning of the school: how to raise the standard of those who are at the lowest level in mathematics and, at the same time, to keep the lessons interesting also for those advanced pupils who may not learn anything new during the two first years. Earlier studies (Metsämuuronen, 2013a; 2017a; Metsämuuronen & Tuohilampi, 2014) indicate that the schooling and supporting system in Finland can turn the wide distribution of performances at grade 1 into a normal (at grade 3) and even a kurtic

normal (at grade 6) after which it, again, widens at grade 9 and even more at the end of secondary education when the national distribution is even wider than what is at the beginning of grade 1.

One part of the challenge is, how to prevent pupils with very low (or even average) level of proficiency in mathematics to fall into an abyss of mathematic anxiety, low self-esteem in mathematics, and underachieving in the studies during the basic education. It may be valuable to try to detect those pupils who have real challenges related to dyscalculia or parallel learning disability related to mathematics. Maybe, at some point, some kind of numeracy screening tests such as functional numeracy assessment (Funa) test (see Funa consortium, 2019; Räsänen et al., 2021) could be used in an early phase, and relevant scaffolding techniques and teaching methods could be developed to help these children during the first stages of development of mathematical concepts, procedures, as well as abstraction and thinking—maybe even before the grade 1.

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# Preschoolers' ways of experiencing numbers

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In this paper we direct attention to 5–6-year-olds' learning of arithmetic skills through a thorough analysis of changes in the children's ways of encountering and experiencing numbers. The foundation for our approach is phenomenographic, in that our object of analysis is differences in children's ways of completing an arithmetic task, which are considered to be expressions of their ways of experiencing numbers and what is possible to do with numbers. A qualitative analysis of 103 children's ways of encountering the task gives an outcome space of varying ways of experiencing numbers. This is further analyzed through the lens of variation theory of learning, explaining why differences occur and how observed changes over a prolonged period of time can shed light on how children learn the meaning of numbers, allowing them to solve arithmetic problems. The results show how observed changes are liberating new and powerful problem-solving strategies. Emanating from empirical research, the results of our study contribute to the theoretical understanding of young children's learning of arithmetic skills, taking the starting point in the child's lived experiences rather than cognitive processes. This approach to interpreting learning, we suggest, has pedagogical implications concerning what is fundamental to teach children for their further development in mathematics.

Keywords: arithmetic, numbers, phenomenography, preschoolers, variation theory

## 1 Introduction

Research on early mathematics education from the last four decades provides multiple observations of children solving arithmetic problems, offering a comprehensive picture of the strategies children use and the common trajectory of arithmetic skills development (see e.g., Baroody, 1987; Baroody & Purpura, 2017; Fuson, 1992). This body of knowledge has influenced many curricula and guidelines for teaching mathematics in the early years (Cross et al., 2009; Sarama & Clements, 2009). What has still not been revealed, however, is what children explicitly learn when developing a more advanced understanding of numbers that becomes useful in their arithmetic problem-solving. This calls for taking an educational perspective in interpreting children's arithmetic skills. In this paper, we aim to contribute to filling this knowledge gap by offering an alternative approach to describing children's

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learning, starting from how numbers appear to them (in a phenomenological sense; see Marton & Neuman, 1990). This, we suggest, has implications regarding what is fundamental to teach for children’s mathematical development. More specifically, we aim to answer the following research questions:

1. How can preschool children’s ways of experiencing numbers in an arithmetic task be described?
2. What distinguishes changes in their ways of experiencing numbers over time?

These questions are answered through empirical research involving 5–6-year-old preschool children solving an arithmetic task characterized as a “missing addend problem”. Through the lens of phenomenography and the variation theory of learning (Marton, 2015), we analyze the children’s encounter with numbers and how their ways of experiencing numbers differ and change over time.

### 1.1 Learning arithmetic skills in the early years

There are many documentations of children’s strategies in arithmetic problem-solving and the trajectory of these strategies. Baroody (1987) is a researcher who is often referenced, describing the development starting with counting skills closely connected to physical countables (enumeration) and consecutively integrating other skills of significance for arithmetic problem-solving (such as the cardinality principle and the succession of numbers on the number line). In line with this way of describing development, when more skills are mastered, this allows the child to make use of mental representations such as the number line, without having to construct the number sequence every time by counting from one, alleviating the cognitive load. Crucial in the development of arithmetic skills is presumably the child’s ability to construct units and an understanding of numbers as compositions of units in a part-whole relationship (Baroody, 2016). Nevertheless, there are empirical observations of children who do not develop these mental representations and continue constructing numbers by counting single units in all tasks they encounter. In other words, they do not invent efficient arithmetic strategies in which number relations (based on the composition of units) can be applied (Neuman, 1987; Ellemor-Collins & Wright, 2009).

The research based on cognitive science primarily emphasizes counting as the foundation for arithmetic development (Baroody, 1987; Fuson, 1992). However, an overemphasis on counting strategies may delay children’s development of more



advanced mathematical skills, according to Cheng (2012, p. 30), because “preschool children who receive continuous encouragement when using counting strategies are reluctant to try the new more advanced decomposition strategy. ..., these children prefer to use such seemly easier and effortless counting strategy”. It has been shown that low-achieving students, even when they are older, often rely on counting strategies, which suggests that counting skills on their own will not develop arithmetic understanding and efficiency in problem-solving (Ahlberg, 1997; Christensen & Copper, 1992; Neuman, 2013). Determining the quantity of a missing part or difference when changes in quantities occur that cannot be enumerated (because missing parts are unknown, not “visible”), particularly in larger number ranges, demands a sense of numbers as constituting a part-whole relationship (Baroody, 2016; Peters et al., 2012).

The need to experience numbers’ part-whole relations was also confirmed in a recent study of preschool children’s arithmetic skills, concluding that children knowing the cardinality of numbers (e.g., the last number word said when counting items one-after-the-other means the whole set of counted items) and ordinality (numbers have internal relations to one another: adding one makes the next number in the counting sequence) is not sufficient for solving even simple arithmetic tasks. It is only when children realize that numbers (in addition to their features of cardinality and ordinality) can be seen as a triad of related numbers that they are able to solve arithmetic tasks or compare sets without concrete countables available (Björklund, Marton et al., 2021). Thus, learning to solve arithmetic problems constitutes the development of a complex of skills that might not be explained by solely constructing mental representations. As we suggest in this paper, the child’s perspective on numbers, and how numbers appear to them, may be a necessary addition to our knowledge of how children learn basic arithmetic skills.

There have been attempts to describe how numbers are understood by children, not least by Piaget (1976/1929), who described intellectual development as qualitative changes in perceptions. His seminal work (Piaget, 1952) concerns how children structure their experiences into knowledge. The structuring process, in Piaget’s view, results in similarities and differences that constitute psychologically real entities. Such a psychological formal structure is assumed to be applicable to different concepts. However, Piaget’s thesis has been criticized for not explaining why a child is then able to express an advanced conception of some phenomenon presented in one situation but fails to do so in another, even though the encountered task or concept

seems to be similar (Smedslund, 1977). This way of describing children's development of number knowledge places the focus on the child rather than the child's experienced world. Starkey and Gelman (1982) present an alternative view to the Piagetian one, proposing that an early understanding of arithmetic is related to particular principles, and that the understanding of these principles proceeds through increasingly complex levels. This view involves the child perceiving not any phenomenon in general but rather numerical phenomena specifically. That is, attention to certain principles is necessary in order for numbers to be understood (see Gelman & Gallistel, 1978). What the research mentioned above focuses on is how knowledge is constructed and transformed in children's development, from less effective toward more effective and valid knowledge.

The interest in understanding children's qualitatively different ways of understanding numbers can be found in several contemporary studies, while taking different theoretical perspectives to interpret what numbers mean to children. For example, Lavie and Sfar (2019) describe the development of children's reasoning with quantities over a prolonged period of time and conclude that number words are indeed commonly used but bear different meaning in children's problem-solving. That is, number words are not necessarily used for enumerating but often instead for estimating and comparing in a sense of "more or less". Already in the 1980s, Neuman (1987) presented a study of children's number knowledge with a strong emphasis on how numbers' meaning appears to them. Based on interviews with school-beginners, she aimed to theorize children's ways of creating concepts of numbers and described this as a trajectory starting from the prenumerical, moving through the early numerical, and ending in numerical concepts. *Prenumerical concepts* are expressed in children's intuitive or learned gestalts of quantities, known as subitizing or recognizing patterns of, for example, two pairs making four, but if separated (spatially) the two pairs would be conceptualized as different. *Early numerical concepts* include several ways of attending to numbers, such as a primitive way of seeing number words as relating to quantities but lacking numerical meaning or making use of number words for "fair sharing", meaning that partitioning is focal to the child in an intuitive sense but the exact quantity (number) is irrelevant; thus, any number word is possible as an answer. Furthermore, some children show an understanding of numbers as "names", which means the number words are seen as names of objects: When adding  $4+5$  the child answers 5, as the fifth object is added, resulting in the last said counting word being the answer. According to Neuman, this

indicates that the number concept is purely ordinal in character and that 5 does not include 4 or any smaller number but is rather a label given to a specific object. Some of Neuman's observations revealed that the size of a quantity is primary to a discrete number of items, a category called "estimates". Number words, then, rather mean "much" or "a little", and there are no computational strategies related to this number concept. Neuman states that these early numerical conceptions are integrated toward an indissociable meaning of cardinality and ordinality, thus leading to numerical concepts. *The numerical conceptions* Neuman found in her studies were "structuring" and "counting", which allow the child to determine the answer to "how many" questions as an exact number of units. Counting is subordinate to structuring, however, as it is important to recognize and also be able to create patterns to represent part-whole relations. If the child's conception of number is restricted to the counting category, according to Neuman's studies this will lead to mathematics difficulties because numbers are then measured only in their smallest single units, which leads to difficulties in keeping track and the cardinal and ordinal meanings of numbers appearing as parallel, leaving the part-whole relation undiscerned. Here, Neuman highlights the theoretical basis of phenomenography, in which learning is regarded as changes in conceptions. For example, incorrect answers to simple arithmetic problems do not imply an absence of learning but can indeed reflect qualitatively different ways of understanding numbers. For instance, regarding "names" and "estimates", which are error-prone conceptions, according to Neuman these are both important parts of children's creation of number concepts that will eventually develop into more advanced number concepts. Similarly, children may very well be able to complete an addition task correctly, but their strategies reveal different conceptions of numbers, of which one (structuring) is a path to development and the other (counting) is not.

What stands out in the research on early numerical learning and development is the (methodological) need to interpret children's actions as expressions of awareness. Ahlberg (1997) clarifies this as two levels of descriptions: Strategies or ways of handling numbers are what can be captured in an observation, but what we need to make interpretations of is what a child is focally aware of in a problem-solving situation and how the child structures this information. How the latter is conceptualized, however, depends on the researcher's theoretical perspective, which is why we sometimes find contradictory explanations of how children learn arithmetic skills (see Björklund, Marton et al., 2021). Ahlberg conducted a study similar to

Neuman's, taking the same theoretical approach and finding similar categories. However, she takes the interpretation one step further, describing children's ways of handling numbers and relating them to their ways of experiencing the meaning of numbers. She concludes (1997, p. 35) that "... using these different ways of handling number children's awareness is directed towards various aspects of them. These different aspects of numbers presented in the children's awareness constitute their understanding and consequently they understand the meaning of numbers in qualitatively different ways". According to Ahlberg (1997), the different ways of understanding numbers are, as: i) number words, ii) extents, iii) position in a sequence, and iv) composite units. These different ways of understanding numbers are explained by what is foregrounded in the child's awareness. In this sense, learning arithmetic skills entails experiencing and simultaneously perceiving these as different aspects of number. However, Ahlberg does not elaborate on how this is executed as a learning process that also includes the mathematical aspects (such as cardinality and ordinality). Even though Neuman and Ahlberg made great efforts to theorize children's understanding of numbers based on the different ways in which children experience numbers, they did not fully come up with a theoretically driven conclusion regarding *how* children come to change their way of experiencing numbers (and thus develop their arithmetic skills).

Regardless of whether one takes a cognitive or phenomenographic approach, children's handling of numbers (their strategy use) is not in a one-to-one relation with a certain way of understanding, even though some clues can be revealed from their actions. In sum, while there is no lack of observations of children using numbers with different meanings, our aim is to contribute theoretically underpinned explanations as to *why differences occur* and how children learn arithmetic skills.

## 2 Theoretical framework

The theoretical lens we apply in our study is phenomenography and variation theory. Phenomenographic research investigates different ways in which the same phenomenon can be experienced by a group of people (e.g., Marton, 1981). Its goal is to find and systematize forms of thought by which people interpret phenomena in their surrounding world. This directs attention to an experiential perspective that highlights individuals and their ways of experiencing (or seeing, perceiving) phenomena they encounter. Phenomenography is a research orientation with the aim to describe, and what it describes is conceptions. "Conceptions" tell what the

phenomenon looks like to the individual (in our case, how numbers appear to the child), and have two intertwined features: the global meaning of the conceptualized phenomenon and a structural feature, which constitute the specific combination of aspects that are discerned and focused on. Thus, a conception (or a certain way of experiencing a phenomenon) is both a holistic experience of a phenomenon and at the same time constitutes a complex of discerned aspects of the same phenomenon (Marton & Pong, 2005). If some aspect that was previously undiscerned is suddenly discerned, this alters the global meaning to the person. Thus, in phenomenographic research, descriptions of conceptions are based on explorative forms of data generation and interpretative character of data analysis, resulting in qualitatively different categories (Svensson, 1997). This means that the results of a phenomenographic investigation comprise a group of persons' knowledge; not in terms of what is considered objectively right or wrong but in terms of the meaning a phenomenon in the surrounding world has for these persons. In recent phenomenographic studies, this focus on describing conceptions is labelled *ways of experiencing* phenomena (Marton, 2015).

The phenomenographic approach has significance for describing and investigating learning, taking its starting point in the meaning that appears to the learner. The phenomenographic research approach has been used for many years to describe students' ways of experiencing different phenomena as a point of departure for understanding why participating in the same teaching situation can result in different learning outcomes (Marton & Booth, 1997). However, it is not enough to learn that children convey different ways of experiencing; in educational studies, it is significant to also know *why* these differences occur. In the phenomenographic approach this is not explained in terms of cognitive deficits, for example, but as being due to differences in how the learning object appears to the children. Even so, in order to explain learning and how to advance the ways the learning object appears to a child, one needs to distinguish what constitutes the different ways of experiencing the learning object.

The main question in variation theory of learning (Marton, 2015) is what constitutes the learning of a specific content. A fundamental idea, based in phenomenography, is that learning entails changes in ways of experiencing a certain content, which is why a central question in the theory involves what the learner needs to "see" that will make this change. Ways of experiencing content constitute the learner's differentiation of aspects of that content (cf., Gibson & Gibson, 1955). The

fundamental principle in variation theory is that the combination of the necessary aspects for handling numbers in an arithmetic task, arrived at by a particular child, defines his/her way of experiencing numbers. When a new (or rather, not previously attended to) aspect is discerned, this liberates a new way of experiencing numbers and thus what the child can do with numbers. In line with this way of reasoning, children's strategy use in arithmetic problem-solving thereby involves expressions of certain ways of experiencing numbers, which in turn is a function of discerned aspects of numbers.

### 3 Methods

To deepen our knowledge of children's learning of arithmetic skills, we studied how numbers are experienced by preschool children and what aspects of numbers appear to them that inform their use of arithmetic strategies. To gain these insights, we conducted interviews with 103 preschool children in their final year of preschool<sup>1</sup>. The interviews were conducted by researchers experienced in educational studies and interviewing children, and were held individually at the children's preschools. Tasks were given orally, and the children were encouraged through follow-up questions to explain how they had come up with their answer. They were also encouraged to use their fingers if they wanted to, but no other manipulatives or tools were offered other than what was part of the task. Nevertheless, some children made use of objects found in the room to support their reasoning.

All the children's legal guardians had given their informed consent for the children to participate in the study. The interviews were video-recorded to allow detailed analyses of the children's actions and utterances. If permission to video-record had not been given, detailed field notes were taken by an assisting researcher. The children participated in the task-based interview on two occasions (8-month interval). The children's mean age was 5 years 3 months at Interview I and 5 years 11 months at Interview II. The participants, from three suburbs outside a large Swedish city, all spoke fluent Swedish and were of mixed socioeconomic backgrounds.

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<sup>1</sup> Preschool is a voluntary pedagogical practice in Sweden for children 0-5 years of age, with a high attendance rate (95% of 5-year-olds the year of the study and 85% of all children aged 1-5).



### 3.1 Data

In this paper we use responses to one task in the interview as our object of inquiry. The task was inspired by the “Guessing Game” task used in Neuman’s (1987) study, in which the interviewer hides a number of buttons in two boxes and asks the child to guess how many there could be in each box. A similar number decomposition activity is the “hidden item task” in Tsamir et al.’s (2015) interview study with 5-6-year-olds, in which seven identical items were used, one set visible in the interviewer’s hand and the rest hidden. The child was asked how many items were hidden in the closed hand. The task was repeated, altering the visible number of items.

Our version of the task, also given orally, includes seven identical glass marbles. The child is initially asked to count the marbles, which are lined up on the table. The interviewer then hides the marbles in her two hands and thereafter the child is asked how many marbles could be in each hand. In the second step, the interviewer opens one hand and lets the child see some of the marbles and asks the child to figure out how many are hidden in the closed hand. After each answer, the interviewer asks follow-up questions to encourage the child to reason about how s/he came up with the answer. The child is given the task three times, altering the partitioning of the seven marbles.

In the analysis we present here, we have selected only one part of the task – the interviewer shows four marbles in her opened hand and the child is to figure out how many are hidden (3) in the closed hand – and only the first round that the task is given. The task corresponds to common “missing addend” tasks in mathematics education, without relying on formal symbolic knowledge, and is thereby suitable for preschool children who have not yet attended formal arithmetic education.

Data for analysis consists of 189 observations of the 103 participating children (92 observations in Interview I and 97 in Interview II). Data was excluded if the child gave no response to the task.

### 3.2 Analysis

To answer our research question, we conducted two consecutive analyses. First, we did a qualitative analysis of the children’s ways of experiencing numbers in the task in both interviews (189 observations in total). The unit of analysis was the observed instances of children’s different ways of handling numbers, shown in both verbal utterances and gestures such as finger patterns. We followed the principles of

variation theory (Marton, 2015); that is, the child acts in accordance with aspects that are discerned at a particular moment, which defines the child's way of experiencing numbers. For example, when shown four marbles in one hand, one child responds "If I add two it only makes six, so it has to be three" and another child "After four comes five, then six and seven, there are seven in the other hand". Considering the first child, we interpret the response as the child experiencing numbers' relation and thereby manages to handle the given part, the missing part and the whole as a cardinal set of composed units. The second child is interpreted to express a way of experiencing numbers as labels given to each item, why it is logical to that child that the last item is "seven", however not expressing a meaning of numbers as composed units and thereby not related to one another in a sense of cardinality. Different acts reflect different ways of experiencing the meaning of numbers. The results from such an analysis are the phenomenographic categories of meaning that appear to the children. This is reflected in our descriptive categories "numbers are experienced as...". These categories present an outcome space of a limited number of qualitatively different ways of experiencing numbers, and this variation is further explained in terms of discerned mathematical aspects. Thus, the analytical process is a constant interchange between interpretations of how numbers appear to the child and what aspects the child seems to discern, as expressed in words and gestures. The children's expressions are sometimes very subtle; the video recordings allowed for reiterate viewing. Each observation has been coded and categorized by two or more researchers, followed by collective discussions within the research group.

Initially, we coded each child's answer according to which numbers they gave as their answer. Thereafter, we categorized the answers into groups with similar answers and compared the children's ways of explaining their answers within each group. In some cases, children who answered with the same numbers were categorized differently as their different ways of experiencing numbers were identified based on their ways of explaining and reasoning about how they had come up with their answer.

What counts as the "same" conception can be expressed in linguistically different ways, and what can be seen as different conceptions can be expressed in similar language (see Neuman, 1987). Thus, interpreting children's conceptions or ways of experiencing numbers is a comprehensive process based on impressions from both verbal and gestural responses. For example: "After four comes three, maybe it's three? You start with five (raising index finger), then comes four (raising middle finger), and then comes three (raising ring finger)". The combination of verbal and gestural

expressions by the child thereby reveals what she discerns (the phenomenon's structural features) and how numbers appear to her (the holistic meaning).

Six categories were found empirically (also reported in relation to other tasks in Björklund, Ekdahl et al., 2021 and Björklund & Runesson Kempe, 2019), and are to some extent similar to previous findings in studies with 6- and 7-year-olds (Ahlberg, 1997; Neuman, 1987). This means, the outcome space of the first analysis partly confirms earlier findings of children's ways of experiencing numbers and partly adds new ones, not described before. In another group of children, it may be possible to find yet additional ways of experiencing numbers (or lack what has been found in our, Ahlberg's or Neuman's studies). The large number of observations do however ensure that our study covers those ways of experiencing numbers that are common among children attending the last year of Swedish preschool.

Second, we selected 90 of the children for whom we had observations from both interviews in order to analyze the changes in their ways of experiencing numbers. This is presented in two ways: on a group level to give an overview of the trajectory of changes, and then on individual case level. The cases are analyzed on a micro-level to gain insights into what in particular constitutes their changed way of experiencing numbers in terms of discerned aspects of numbers. This micro-analysis contributes to our understanding of what the children actually learn to discern that changes their way of experiencing numbers.

## 4 Results

We present the results from our analysis in three sections: First, we describe the ways of experiencing numbers that appear in the empirical data. Second, we present changes in ways of experiencing numbers within the group of children, and third, we illustrate how changes are expressed empirically on an individual case level.

### 4.1 Ways of experiencing numbers

From all of the observations in both interviews, we find six categories of qualitative different ways of experiencing numbers that impact the children's strategies in completing the Guessing Game (see Table 1). Differences between categories appear in terms of discerned aspects of numbers, but there are also differences within each category in terms of how the discerned (and undiscerned) aspects are coming through in the children's acts and utterances.

**Table 1.** Ways of experiencing numbers expressed in the Guessing Game.

<b>Numbers experienced as:</b>
A. Numbers as Words
B. Numbers as Names
C. Numbers as Extent
D. Numbers as Countables
E. Numbers as Structure
F. Numbers as Known Facts

#### 4.1.1 A: Numbers as Words

Number words are used without having any meaning of cardinality or ordinality. Children know that number words represent a certain category of words that are used in situations in which groups of items are handled. They use random number words either solely or in a random sequence or repeat a counting word from the given task. In the Guessing Game we observed this way of experiencing numbers among children who answered with random number words, such as Kevin: “Five, seven, thousand”. Even though the moment before the child counted, or at least recited, the counting sequence while pointing at the marbles one-to-one, there is no numerical relation foregrounded in the child’s utterance when asked how the marbles may be partitioned. In the task, the number of objects also exceeds the subitizing range, and as the child does not discern numbers’ cardinality or ordinality, counting to determine quantities is not an option – it is a procedure you use when asked “how many”, but the number words used do not have the meaning of a composed set.

#### 4.1.2 B: Numbers as Names

When experiencing Numbers as Names, number words are ordered in a sequence and thereby have some relation to each other in terms of ordinality. In this sense, number words can describe “the  $n^{\text{th}}$ ” object, as in an object following another object. However, there is no cardinal meaning involved, as in a consecutive word meaning “one more”. This has significant impact on how numbers are used and how a numerical task is encountered. Otto answers by first counting and pointing at the visible marbles – “One, two, three, six” – and then tapping on the knuckles and back of the interviewer’s closed hand: “One, two, three, four, six. Seven”. Otto’s actions indicate that numbers appear as single objects that are labelled with number words. He never answers with

one word, as in labelling a collection, but always counts on the sequence starting from one. Another expression of experiencing Numbers as Names is observed when children answer with two consecutive number words: When one hand with four marbles is shown, Lydia confidently answers “Five there”, pointing at the closed hand. Giving two consecutive number words as an answer to how many objects there might be in the closed hand is a quite common response, even when the child confirms that there were seven marbles on the table from the start. The ordinal meaning appears in the foreground, for example by Sanna: “After four comes three, maybe it’s three? You start with five (raising index finger) then comes four (raising middle finger), and then comes three (raising ring finger)”. When experiencing Numbers as Names, the number words are closely connected to objects that are to be enumerated, which is why the words rarely exceed seven because the counting sequence and ordinality are foregrounded – the child labels objects starting from “one”. This sometimes leads to children answering “seven” when they see four marbles in the opened hand, even though they without difficulty enumerated the set of marbles to be seven when seeing them all on the table. When ordinality is foregrounded (and cardinality undiscerned) this makes sense to the child, as the marbles labelled one, two, three, and four are indeed visible in the opened hand and the marble known as “the seven” then has to be in the enclosed hand. When numbers are experienced as names, these cannot be added or subtracted from other names. Because the cardinal meaning is undiscerned, number words can not be seen as parts of a larger collection labelled with another word (or: the child is unable to see that four is part of the larger set, seven). Some children, like Malik, make attempts to operate with the names “After five comes four, after four comes three, it might be three there”, which indicates that the counting sequence supports him in maintaining attention on objects that are to be enumerated but are hidden in the interviewer’s closed hands (see Category D for advancements resembling of this way of operating with the counting sequence).

#### 4.1.3 C: Numbers as Extent

When numbers are experienced as Extent, they have an approximate value that indicates that a cardinal meaning is discerned. The ordinality of numbers is not discerned, and the relation between numbers is limited to “more or less” in an undistinct meaning, like Agnes: “Perhaps a little bit more than these (pointing to the opened hand with four marbles)”. In the Guessing Game, this way of experiencing Numbers as Extent is observed when children give answers characterized by some

sense of plausible quantities related to the task. For example, Jamila says “I don’t know how many there are, I have to look to know, but I think three”. Some children give answers that are close but not correct. Characteristic of these instances is that the child does not give a reason for the answer or express his/her way of coming up with it; thus, there is no explicit relation between the numbers discerned that would enable the child to reason about why a certain number is a plausible answer. In cases in which the children motivate their answers they are described as guesses, which is likely because the lack of discerned ordinality hinders any proper operation with the numbers in the task. When children attempt to reason their way to an answer it is often directed at equality in their partitioning, such as Olivia suggesting “doubles”: “Equally many as in the first one [opened hand]”.

#### 4.1.4 D: Numbers as Countables

In some children’s ways of experiencing numbers, we see a strong influence of the ordinal aspect of number and some idea that numbers can relate to each other. The child discerns numbers constituting a set of items, thus having a cardinal meaning as well, but this set is experienced as added units of “ones”. There is a clear difference to Category B, because here numbers are not connected to specific items but rather discerned as single units in themselves, which can be counted. Due to the dual meaning of numbers (cardinality and ordinality), it is possible to add and subtract by enumerating (and thus creating) sets in what is commonly known as the “counting all” strategy. William, for example, makes a finger pattern of four by raising and counting one finger at a time, then raising fingers on the other hand while counting all raised fingers from one, ending up with seven raised fingers together (four on one hand and three on the other), and then counting the last three raised fingers on the other hand. That is, he operates with the known numbers by representing them on his fingers but experiences them as added single units and has to create the numbers starting from one. It then becomes difficult to relate numbers to each other; they have to be operated on directly, and re-created, to be perceived. Another expression of this way of experiencing numbers is shown by Liam when answering: “Maybe there are five (pointing at the closed hand). Because there can be four, five, six. And seven, eight.” The last utterance indicates that the numbers constitute countable (single) units: He counts on the counting sequence, and then counts or perceives how many number words were said.



#### 4.1.5 E: Numbers as Structure

Experiencing Numbers as Structure is based on the child's discernment of numbers constituting composite sets or units, which may simultaneously be related to other units in a part-part-whole relation. In the Guessing Game we observe that children sometimes use finger patterns to represent numbers, particularly to structure numbers' parts and whole. This leads to their operating on the relation between parts and/or the whole simultaneously and finding the missing number (hidden marbles) through arithmetic reasoning (this differs from Category D, Numbers as Countables, in which numbers are single units constituting a set and adding two sets means that a new set is created from the single units of the two earlier ones). There are several actions that this way of experiencing numbers opens up for. One is shown by Sara, who creates some of the units by counting, "counting on", taking as a starting point the given number (4), keeping the whole (7) in mind, and adding on (3) by raising one finger at a time until the finger pattern seven is visible. Also, without fingers as an aid for structuring numbers, we can see the same way of experiencing numbers in children's reasoning toward their answer, such as Alex: "If I add two it only makes six, so it has to be three." The difference to Category D here is that the child simultaneously keeps the parts and the whole in the foreground, thus relating and reasoning about the four being part of the larger seven, like Mary: "One, two, three, four (raising one finger for each number word, then simultaneously showing two more fingers on the other hand, folding down the first four raised fingers and raising the fifth finger, now holding the two fingers and the single finger close together) three!" Seeing numbers in this way can also be observed, for example, when children start by showing a finger pattern on one whole hand and the thumb and index finger on the other, then moving the thumb on the whole hand to make a gap between the rest of the (four) fingers and thus creating a unit of the thumb and the two on the other hand, showing four, three, and seven at the same time, in this case not created by counting but rather by recognizing the units that the fingers represent.

#### 4.1.6 F: Numbers as Known Facts

Experiencing Numbers as Known Facts means that children instantly recognize numbers as a part-whole relation; that is, numbers can be partitioned in different ways, and smaller numbers are parts of larger ones. This is shown when children give an instant (correct) answer. Most children also explain their answer in terms of retrieved facts, like Christa: "Because three and four make seven". This category

differs from Category E, Numbers as Structure, in that the children do not compose and decompose numbers, for example on their fingers or verbally reason their way to an answer, but simply “see” the number relations.

## 4.2 Changes in ways of experiencing numbers on group level

In the following, we describe how children’s ways of experiencing numbers (see the six categories described above) change over time on a group level. [Table 2](#) gives an overview of how many observations were found within each category in Interviews I and II, only including children who responded to the task in both interviews (n=90).

A comparison between the two interviews shows that the changes are mainly positive. Categories A-C, which involve ways of experiencing numbers that do not impose any operations based on numerical features except for guessing and intuitive estimations of the size of the amount, dominate the first interview (84.4%) but have decreased to 24.5% in the second one. In both interviews, Category D is rare. Categories E and F, which express an awareness of number relations and open up for children to operate with numbers as part-whole relations, are also quite rare in the first interview but in fact dominate in the second one (74.5%). This means that, over the course of one preschool year, the children in general have changed from prenumerical to numerical ways of experiencing numbers and are consequently able to solve the Guessing Game using arithmetic strategies when they finish their last preschool year.

**Table 2.** Children’s ways of experiencing numbers in Interviews I and II (N=90).

Note: Percentages do not add up to 100% due to rounding error.

Category	Interview I		Interview II	
	Frequency	Percent	Frequency	Percent
A. Words	4	4.4	0	0
B. Names	28	31.1	8	8.9
C. Extent	44	48.9	14	15.6
D. Countables	2	2.2	1	1.1
E. Structure	2	2.2	26	28.9
F. Known Facts	10	11.1	41	45.6
Total	90	100.0	90	100.0

In **Table 3** we see how the changes in ways of experiencing numbers are distributed. There seems to be a hierarchy in the distribution, with all but five observations moving in the direction A toward E. The five exceptions involve four observations of children expressing their experience of Numbers as Extent (C) in Interview I and Numbers as Names (B) in Interview II, and one child expressing Numbers as Known Facts (F) in Interview I and Numbers as Structure (E) in Interview II. Ten children remain in the same category (4 in B and 6 in C). Nine children had already expressed Numbers as Known Facts in Interview I.

**Table 3** further shows that there is a difference in how ways of experiencing numbers develop toward Categories E and F; that is, an awareness of numbers' part-whole relations that leads to opening up for children to complete the arithmetic task. Experiencing Numbers as Words (A) or Names (B) is found to change into experiencing Numbers as Structure or Known Facts among 21 of the children (22.2%), while children who experience Numbers as Extent (C) or Countables (D, however rarely observed) more often (38.8%) develop into the advanced ways of experiencing numbers (E and F).

**Table 3.** Transition between the categories, Interview I to II, N=90.

		<i>Interview I</i>						
		A. Words	B. Names	C. Extent	D. Count.	E. Structure	F. Known Facts	<i>Total Int. II</i>
<i>Interview II</i>	A. Words	-	-	-	-	-	-	-
	B. Names	-	4 (4.4%)	4 (4.4%)	-	-	-	8 (8.9%)
	C. Extent	2 (2.2%)	6 (6.7%)	6 (6.7%)	-	-	-	14 (15.6%)
	D. Countables	-	-	1 (1.1%)	-	-	-	1 (1.1%)
	E. Structure	2 (2.2%)	9 (10.0%)	12 (13.3%)	2 (2.2%)	-	1 (1.1%)	26 (28.9%)
	F. Known Facts	-	9 (10.0%)	21 (23.3%)	-	2 (2.2%)	9 (10.0%)	41 (45.6%)
	<i>Total Int. I</i>	4 (4.4%)	28 (31.1%)	44 (48.9%)	2 (2.2%)	2 (2.2%)	10 (11.1%)	90 (100.0%)

Note: Percentages do not add up to 100% due to rounding error.

### 4.3 Changes in ways of experiencing numbers on individual case level

Table 3 shows that the categories Numbers as Names (B) and Numbers as Extent (C) were the most common ways of experiencing numbers in the first interview. In the second interview, most children in these two categories were categorized as experiencing Numbers as Structure or as Known Facts. In this part, we illustrate this change by analyzing four children's ways of experiencing numbers on the two interview occasions and particularly the changes that have occurred.

#### 4.3.1 From Numbers as Names to Numbers as Structure

The change from experiencing Numbers as Names to Numbers as Structure is significant to the child's learning of arithmetic skills, because of the foregrounded cardinality and number relations that appear in the child's awareness in the latter category. It seems critical that the child discerns how number words label not the concrete objects but rather a set that can be composed of any objects. This change in how numbers appear to the child opens up for relating sets to each other for a comparison of quantities, but also how sets (and thus numbers) relate in a part-whole fashion.

The example of Mary will illustrate the specificity of the change from experiencing Numbers as Names in the first interview to experiencing Numbers as Structure in the second one:

*Excerpt 1: Mary, Interview I*

I: (shows four marbles in her opened hand) If there are four there, how many are there in this [closed] hand?

Mary: Seven.

In Interview I, when Mary is shown four marbles in one hand and asked how many are in the other, she answers that there are "seven" in the other hand. This answer is typical of the children who experience Numbers as Names. We interpret her answer as an illustration of her discerning the ordinal but not yet the cardinal aspect of number – the marbles are labelled with number words, making the answer "seven" perfectly logical, as the seventh marble is indeed hidden in the interviewer's hand. The lack of discerned cardinality meaning comes through in that she did count the marbles one-by-one before starting the game, but her "seven" does not constitute a set of seven items (if so, she would realize that there cannot be a set of seven marbles hidden when she sees four in the opened hand).

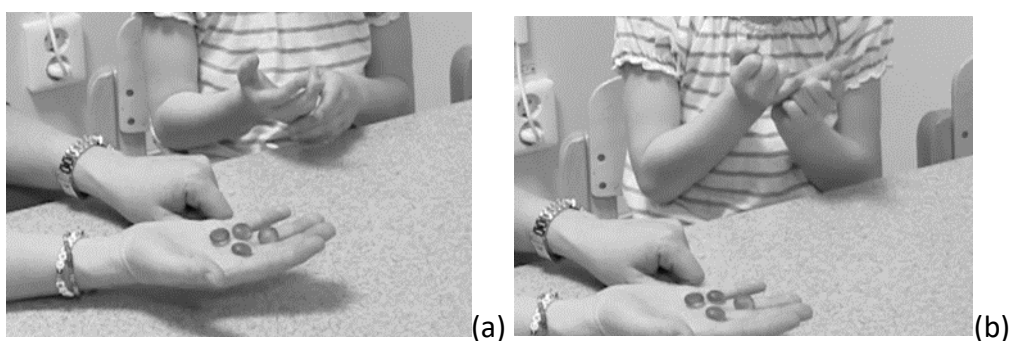
In the second interview, Mary approaches the game in a quite different way, using her fingers to structure numbers in a relationship of a whole and its included parts:

*Excerpt 2: Mary, Interview II*

Mary: Four. (Instantly identifies that there are four marbles in the opened hand.)

I: There were four there. If you now know there are four there, how many are there here then?

Mary: Okay. (Raises four fingers one at a time on one hand (Picture a) and then adds three fingers, showing one whole hand and two fingers. Thereafter, she holds up only the three added fingers to the interviewer (Picture b).) Three.



Mary handles the task in a way that shows her experiencing numbers in a more comprehensive way than before, now discerning more aspects of numbers, which allows her to handle numbers differently. Numbers are no longer names labelling objects, as she represents the marbles, even the hidden ones, on her fingers: Four, seven, and three thereby have cardinal meaning for her (see Picture a, in which she makes a pattern of four fingers) and not only ordinal (the fourth object). Thus, the number words do not address the marbles per se, but rather the representatives (fingers) that she is able to structure in order to determine the number of the hidden set of marbles. She does this by first structuring the numbers on her fingers, by which she discerns the relationship between the numbers, seeing four and the hidden part three in the total of seven (see Picture b). The difference between the first and the second interview is that in the second one Mary shows that she has now discerned not only ordinality but also cardinality, as well as the part-whole relation of numbers, and is able to keep these aspects foregrounded at the same time in order to complete the arithmetic task. She is also able to see units within units, for instance when she sees that one finger on her right hand and the two fingers on her left hand make a new unit of three ( $3/1/2$ ), an additional indication of her experiencing Numbers as Structure.

Another example of a similar change in ways of experiencing numbers, but expressed somewhat differently, is done by Clara. In the first interview, she answers that there are “five” in the closed hand. This is interpreted as her likely experiencing Numbers as Names, as answering with a consecutive number indicates that ordinality is in the foreground of her awareness:

*Excerpt 3: Clara, Interview I*

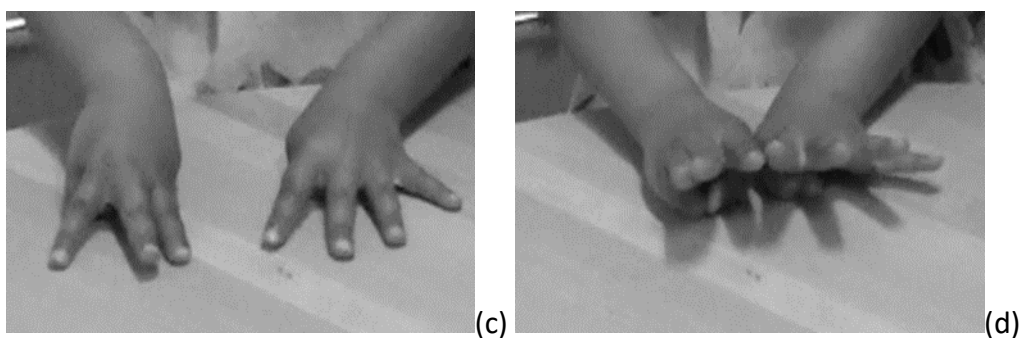
- I: (Opens her hand with four marbles) How many is this?  
 Clara: (Counts the marbles, pointing at them one-by-one) One, two, three, four.  
 I: Four. How many do you think there are in that hand, if there are four there?  
 Clara: Five, I think.  
 I: (Opens her other hand, showing three marbles).  
 Clara: Three!

A possible interpretation of how the numbers appear to Clara is that “five” represents a partition of an imagined number line up to seven, whereby five serves as a limit between “the four” and “the five”, in an ordinal sense. To some extent she is able to discern that a set of marbles comprises “three” in number, but this is isolated from any awareness of sets related to other sets. Thus, she can answer the “how many” question by either counting one-by-one or subitizing small sets but does not yet discern any number relations. This results in her experiencing numbers as isolated units and necessary to set to the concrete objects, or as in Excerpt 3 above, the counting sequence as an order of number words. This indicates, however, that numbers can be represented in an orderly fashion, which is indeed an important aspect to discern, but is not sufficient for forming a way of experiencing numbers that enables arithmetic reasoning. This, on the other hand, is something we can see evidence of in Interview II:

*Excerpt 4: Clara, Interview II*

- Clara: Four there, and three there.  
 I: Now I’m curious. Why do you think there are four there and three there?  
 Clara: Because four plus three is seven (Models three and four on her fingers (Picture c) and puts the fingers together to show seven (Picture d)).  
 I: Okay, let’s check. There were four there. How many are there there then?  
 Clara: Three.





In the second interview Clara instantly answers that there are three in the closed hand and says “Because four plus three is seven”. She structures the numbers on her fingers to show the interviewer how three and four together literally make seven, even though her instant answer indicates that she experiences the number relation as Known Number Facts.

The change in her ways of experiencing numbers is shown in what Clara is able to do with numbers. In the second interview she is able to show why three is the missing part by structuring seven on her fingers, showing the parts and the whole simultaneously. She sees the numbers involved as composite sets, and with this shows that she has discerned cardinality. She has also discerned ordinality, as she simultaneously relates numbers to each other in accordance with the counting sequence, adding smaller units to make the whole seven. Her way of moving the represented numbers (finger patterns) together is one way of structuring numbers that shows her awareness of the part-whole relation. That is, seeing how three and four are both parts of seven in a structural way is a powerful advancement from her earlier way of experiencing numbers.

#### 4.3.2 From Numbers as Extent to Numbers as Structure

In the first interview, many children expressed their experiencing Numbers as Extent, which indicates that the ordinality aspect of numbers is undiscerned. These children do seem to have a sense of numbers’ manyness, but due to the absence of ordinal meaning they cannot organize numbers or sets according to quantity other than in an approximate sense. Consequently, they do not have any repertoire for operating with numbers, either to determine exact quantities, for instance in comparison, or to find a hidden or missing set. Nevertheless, they experience that numbers are related to “more or less”, which allows them to make guesses when asked “how many”.

Sofie is categorized as experiencing Numbers as Extent in the first interview:

*Excerpt 5: Sofie, Interview I*

- I: What do you think?  
 Sofie: (Looks around in the room) As many as, the cookies.  
 I: As many as the cookies there. How many cookies are there then? (Sofie brings the cookies to the table and places them in a group).  
 Sofie: (Points at each cookie) One, two, three, four. Wait (Counts again) One, two, three, four, five.  
 I: Five, you think there are five.

In the first interview Sofie counts the marbles in the opened hand, thus having an idea of number words used in a procedure in which you point and say words in a consecutive order. However, when asked how many marbles there are in the other hand, she makes no attempt to account for the already visible ones, as related to the unknown set; instead, she looks around the room and at a bookshelf with toys near the table. We infer that she experiences some sense of cardinality, as she expresses quantity by saying “as many as the cookies”. As she does not discern any (numerical) relation between the set of marbles and the set of cookies, figuring out the quantity of a hidden set of items is not possible. The change in her way of experiencing numbers in the second interview is apparent, as she then clearly discerns exact numbers and relates them to each other in completing the Guessing Game:

*Excerpt 6: Sofie, Interview II*

- I: How many are there in that one?  
 Sofie: Four (Shows four fingers on her right hand, then on her left hand raises the little finger, and immediately after this the thumb and index finger simultaneously). Three. (see Picture e)



(e)

Sofie immediately sees that there are four marbles in the opened hand, without counting. When asked how many marbles there are in the other hand, she shows a finger pattern of four and thereafter identifies the missing part as a set constituted of

one finger on the right hand and two more fingers on the left hand (in the same way as Mary does, above). She sees the numbers as composite sets that have cardinal meaning. She is able to compose a unit of three, from one finger on the right hand and two fingers on the left, indicating her discerning of the relation between and within the numbers and thereby having developed her way of experiencing Numbers as Structure.

## 5 Discussion

In this paper we set out to describe preschool children's ways of experiencing numbers in an arithmetic task and what might distinguish changes over time. We have approached these questions by suggesting that the child's perspective on numbers and how numbers appear to them may be a necessary addition to our knowledge of how children learn basic arithmetic skills. Our qualitative analyses resulted in six different ways of experiencing numbers, distinguished by which aspects of numbers are discerned by the children. From a longitudinal perspective, we have shown how children's ways of experiencing numbers change and, more specifically, which aspects become critical to discern in order to develop arithmetic skills. Some categories presented in this paper are comparable to previous findings in studies with 6- and 7-year-olds (e.g., Neuman, 1987; Ahlberg, 1997). Particularly Ahlberg's theorizing ambition has influenced the current study, that different ways of handling numbers mean that children's awareness is directed at various aspects of numbers and that this constitutes their understanding of numbers. What Ahlberg did not determine in her research was *how* different ways of handling numbers are connected to ways of experiencing numbers and specifically discerned (or not discerned) aspects. Our study may contribute to fulfilling this ambition by specifically pointing out the difference in how children handle numbers depending on their discerning ordinality, cardinality or both of them and number relations simultaneously.

Earlier studies (Björklund, Marton et al., 2021) have shown that which aspects of numbers children discern is linked to their repertoire of arithmetic strategies. Some strategies, according to the large body of research in the field, are known to be error-prone, such as counting single units if it is the only strategy used by the student (e.g., Ellemor-Collins & Wright, 2009; Neuman, 1987). In our study, we rarely see any counting-based strategies among the preschoolers in either Interview I or II. This could be taken as an indication that it may not be necessary to introduce counting-based strategies in early arithmetic education, as children obviously do not need to

experience Numbers as Countables at any particular point in time; they seem to be able to discern number relations and coordinate cardinality and ordinality meaning in numbers and thus learn to experience Numbers as Structure anyway. Or as Neuman (1987) would describe it, go from non-numerical to numerical conceptions without numbers appearing as countables. A consequence, then, would be that the children do not risk getting stuck in single-unit counting strategies, but instead appropriate numbers as constituting composite sets that can be de-composed and re-composed as means in solving arithmetic tasks (e.g., Cheng, 2012). Taking one's starting point in the child's lived experiences rather than cognitive processes, the key might thus be not to primarily attend to children's skills or abilities (such as frequency of using a certain strategy) but rather to focus on how numbers appear to them and support their discerning aspects that emphasize numerical units and relations. Our analysis of the variation in ways of experiencing numbers supports Neuman's suggestion that children's errors or success in completing arithmetic tasks may be induced by different ways of experiencing numbers; that is, experiencing Numbers as Words, Names, or Extent reflect very different ways of seeing numbers, while the result of completing a task may be the same number word. It is therefore necessary to highlight what appears as focal in the child's way of experiencing numbers, in order to fully understand what (aspects of numbers) are critical for children's ways of experiencing numbers to change into conceptions that allow for more powerful strategies to be used. For example, children who experience Numbers as Extent or Countables are in our study seen to more often develop more advanced ways of experiencing numbers as structure or known facts. This needs though to be the object of further inquiry, examining whether it indicates that experiencing Numbers as Extent is the path to more advanced ways of experiencing numbers or if it is merely an effect of a larger number of observations among this particular group of children. However, according to our observations we can draw the conclusion that during their last preschool year the majority of the children do learn to discern cardinality and ordinality as well as numbers' part-whole relations.

Observing children answering "how many"-questions, for example with random number words or irregular counting sequences, is not new; Fuson (e.g., 1992) and others have presented similar observations among preschoolers in several studies. What we wish to add to this field of research, however, is interpretations of what numbers mean to the children, how numbers appear to them. This would help us understand why children answer with random numbers when the moment before they

were able to point and recite the counting sequence and repeat the last uttered counting word as an answer to “how many”. At the beginning of this paper, we claimed that there is a lack in the field of knowledge concerning what it explicitly is that children learn when they develop a more advanced meaning of numbers, which consequently leads to their using powerful strategies in arithmetic problem-solving. This is partly discussed in Björklund, Marton et al. (2021) in terms of children needing to learn to discern certain aspects of numbers. What the current paper contributes in addition to this is how the discernment of some (but not all) aspects of numbers constitutes a variation in ways of experiencing numbers. This study of ours is theoretically grounded in phenomenography and variation theory. This leads to an emphasis on the emergence of “conceptions”, or ways of experiencing some phenomenon. This means that we use the theoretical framework to describe what the “numbers” phenomenon looks like to the individual, determined by both the global and structural meanings of the conceptualized phenomenon. In line with this, we have intended to describe the variation in ways of experiencing numbers (that is, the global meaning appearing to the child) and how a certain way of experiencing numbers is constituted (that is, the structural meaning of the phenomenon of numbers). The combination is our theoretical contribution, which adds to what, for example, Neuman (1987) and Ahlberg (1997) described and theorized some decades ago.

The connection between discerned aspects of numbers and the way of experiencing numbers that is highlighted throughout the current paper is not only a theoretical contribution. We suggest that it is a key to early mathematics education, as it offers an explanation of children’s different ways of encountering arithmetic tasks and what they need support in discerning in order to develop their ways of experiencing numbers. In particular, it becomes evident that experiencing numbers as composed units that can be related, composed and de-composed is an essential aspect to discern in order to develop arithmetic skills, as shown in the empirical examples. Thus, what educational practices should facilitate is opportunities to explore and experience numbers as representing composed sets. What aspects children discern may be difficult to “see”, but how children experience numbers’ meaning might be the entrance point to understanding their knowledge and skills, as well as what support they need in learning to discern critical aspects.

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# Developing mathematical problem-solving skills in primary school by using visual representations on heuristics

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Developing students' skills in solving mathematical problems and supporting creative mathematical thinking have been important topics of Finnish National Core Curricula 2004 and 2014. To foster these skills, students should be provided with rich, meaningful problem-solving tasks already in primary school. Teachers have a crucial role in equipping students with a variety of tools for solving diverse mathematical problems. This can be challenging if the instruction is based solely on tasks presented in mathematics textbooks. The aim of this study was to map whether a teaching approach, which focuses on teaching general heuristics for mathematical problem-solving by providing visual tools called Problem-solving Keys, would improve students' performance in tasks and skills in justifying their reasoning. To map students' problem-solving skills and strategies, data from 25 fifth graders' pre-tests and post-tests with non-routine mathematical tasks were analysed. The results indicate that the teaching approach, which emphasized finding different approaches to solve mathematical problems had the potential for improving students' performance in a problem-solving test and skills, but also in explaining their thinking in tasks. The findings of this research suggest that teachers could support the development of problem-solving strategies by fostering classroom discussions and using for example a visual heuristics tool called Problem-solving Keys.

Keywords: mathematical problem-solving, heuristics, proportional reasoning

## 1 Introduction

During the primary school years, students develop their understanding of concept of numbers and fluency in arithmetic skills (FNBE, 2016, p. 307). Learning mathematical procedures is important, but it is also crucial to equip students with strong problem-solving, reasoning, and thinking skills (e.g. Lester, 2003; Pehkonen et al., 2013) to give tools for functioning in a complex, unpredictable future. Mathematical problem-solving requires skills to apply variety of different solution strategies and models (Leppäaho, 2018, p. 374). It is not uncommon that while students may excel on routine exercises (those that they have already seen and practiced), they fail to solve problems that differ from those they have previously encountered (OECD, 2014).

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Traditional teaching approaches often focus on learning mathematical facts and procedures. Teachers could take advantage on creating learning environments, which engage students in investigating problems and seeking solutions in an active manner. (Pehkonen et al., 2013, 13.) Näveri et al. (2011, p. 169) point out that if teachers rely on using routine tasks in mathematics lessons, also the learning of students stays on the routine level. Mathematical thinking skills can be developed via problem-solving (e.g. Schoenfeld, 1985; Lester, 2003, Leppäaho, 2018), and on the other hand, problem-based teaching methods can be used to foster deeper understanding.

The importance of developing mathematical reasoning and problem-solving skills is also recognised in international assessments, such as PISA and TIMSS. In PISA the problem-solving competence is defined as “an individuals’ capacity to engage in cognitive processing to understand and resolve problem situations where a method of solution is not immediately obvious” (OECD, 2014, p. 30).

As Leppäaho (2018, p. 368) points out, mathematical problem-solving is learned only by practising it repeatedly. Mathematics can actually be taught *through* problem-solving (see for example Schoenfeld, 1985; Hiebert, 2003; Lester, 2013). This teaching method enables students themselves to engage with meaningful, rich problem tasks and instead of superficial procedure-learning, develop understanding of mathematical concepts and methods. Students should have possibilities to explore a variety of different and unfamiliar problems, even though they would not yet master certain methods or algorithms (Goldenberg et al., 2003, p. 28).

Developing students’ mathematical thinking and problem-solving skills have been flagged as important goals in Finnish National Core Curricula for basic education (FNBE 2004; FNBE 2016). Students should be guided not only to solving problems, but also finding and modifying them (FNBE, 2004, p. 158). According to the mathematics curriculum in Finland, instruction should “support the development of the pupils’ skills in presenting their mathematical thinking and solutions to others in different ways and with the help of different tools” (FNBE, 2016, p. 307).

Expressing mathematical ideas and justifying thinking can be challenging for primary-school aged students, but as Finnish Curriculum (FNBE, 2016, p. 306) underlines, it would be important to learn to communicate ideas and collaborate with peers. Collaborative problem-solving situations, identifying and discussing ideas and participating in explanation-building discourse can help learners in developing their thinking skills (Scardamalia & Bereiter, 2014, p. 3). Collaborative problem-solving situations are excellent opportunities to explore also complex problems, because

different examples and explanations by group members enable better understanding (Sears & Reagin, 2013).

In this research, fifth grade students were introduced general heuristics, which were understood to serve as stepping stones in solving non-routine mathematical problems. At the beginning of the school year, it appeared that many students seemed to struggle in mathematical tasks and especially in explaining their problem-solving processes in written form. Students were introduced to concrete tools called Problem-solving Keys, which were modified from Strategy Keys based on work by Herold-Blasius (2021). The aim was to provide students with a visual reminder of heuristics for mathematical problem-solving tasks. Similar heuristics were outlined also in the Singaporean Mathematics Syllabus 2013 and used as a reference when classifying and modifying the Keys for teaching purposes in Finland (Kaitera, 2021).

The research aimed to map fifth graders' skills and strategies before and after the intervention, which was designed to offer wide variety of mathematical problems and techniques to solve them. The interest was in finding out if the problem-oriented teaching approach influenced on how students solved mathematical problems, which required proportional reasoning.

This research aimed to answer the following questions:

1. What kind of influence did teaching approach, which focused on mathematical problem-solving, have on students' general performance in proportional reasoning tasks and abilities to explain thinking?
2. What kind of differences appeared in students' use of erroneous and correct problem-solving strategies between pre- and post-tests?

The study outlines possibilities to develop mathematics teaching towards a direction, in which students become more active participants in learning process and develop their mathematical problem-solving skills. Another aspect was to answer the 21st-century demands for analysing the teaching practises and creating knowledge as a practicing teacher (see for example Niemi & Nevgi, 2014). The study includes features of a teaching experiment and in this report is referred to as such.

## 2 Theoretical framework

### 2.1 Mathematical problem-solving: focus on heuristics

In many countries, mathematics curricula emphasize the importance of exploring versatile problem-solving activities. These have been a part of mathematics classrooms for a long time, but there is still confusion on what it means in practice. Teachers often understand it as solving word problems (e.g. Lester, 2003; Näveri et al., 2011), or even solving simple, routine arithmetic tasks presented in mathematics textbooks (Näveri et al., 2011). In this study, students were provided with non-routine tasks, which require skills to devise and implement a plan (Polya, 1945/1973) and combine previously learned solution strategies in a novel way (Lester, 2013; Leppäaho, 2018).

An ability to solve mathematical problems in different contexts is an important skill, which can, and should be taught at schools. To be able to invent and test strategies, students need to have basic skills and understanding of problem-solving processes. As Leppäaho (2018, 374–375) points out, in addition to mathematical skills (e.g. how students can use different strategies), for example motivational aspects and reading and writing skills play important roles in an individual's capacity in mathematical problem-solving situations.

Mathematical problem-solving techniques are often called heuristics (Polya, 1945/1973; Schoenfeld, 1985; Goldenberg et al., 2003). Heuristics can be described as non-rigorous, general suggestions for strategies, which can be helpful when solving different types of problems. Learning these techniques and becoming familiar with different problem-solving methods helps students to tackle mathematical problems also in unfamiliar contexts.

Heuristics were linked to everyday teaching by Polya in his book “How to solve it” (1945). Polya outlined a simple four-step problem-solving process, and the following phases are often referred to when defining heuristics:

1. Understanding the problem: what is being asked? What is known, what is unknown?
2. Creating a plan for solving the problem, considering whether the type of the problem is already familiar, choosing the most appropriate heuristic.
3. Solving the problem by carrying out the plan and assessing whether the steps are correct.

4. Looking back and checking if the answer makes sense. (Polya, 1973, 5–6.)

Important first steps of understanding a problem and choosing methods for solving the task are often forgotten when describing elements linked to mathematical problem-solving (Näveri et al., 2011, p. 169). School mathematics often emphasizes teaching certain algorithms to fit certain types of problems instead of providing a wider variety of general tools for problem-solving (Näveri et al., 2011; Leppäaho, 2018). Another important aspect linked to problem-solving can be derived from Polya's views: he outlined the function of the last phase as not only reviewing the process but also discussing it (1973, p. 6).

Heuristics are not the same as algorithms: they rarely prompt a solution, while carrying out an algorithm, which is suitable for a certain type of mathematical problem, leads to a rather unambiguous solution. According to Polya (1973, p. 113), heuristics cannot be used as a tool for rigorous proof. Instead, heuristics belong to a problem-solving process as a part of it. Whereas algorithms are usually constructed of certain predetermined steps, heuristics involve a decision-making process. Students make assumptions on whether a certain approach would work or not and try out different ways to implement the method: for example, in this study, making first a diagram or table provides numerous chances to proceed in solving the problem.

Heuristics can be learned and practiced (Schoenfeld, 1985; Bruder & Collet, 2011) and are generally more applicable in different types of mathematical domains and problems than plain algorithms. Due to the nature of transferability, learning heuristics also supports the development of confidence in mathematical problem-solving (Goldenberg et al., 2003). The aim of teaching mathematics through problem-solving is to equip students with skills to apply previously learned techniques in non-routine and novel situations (Leppäaho, 2018, p. 379).

Polya's four-step model is still useful in today's mathematics classroom and was referred to as a framework to underline different phases of problem-solving; mathematics is more than just filling in the textbook, it could be understood as an activity. Devising a plan and choosing the most appropriate heuristic were supported by visual tools called Problem-solving Keys, which are introduced in Chapter 3.2.



## 2.2 Proportional reasoning as a problem-solving domain

Fifth graders' problem-solving skills were mapped by proportional reasoning tasks. It is an excellent domain to solve mathematical problems linked to everyday life. For example, adjusting the recipe, preparing juice from a concentrate, calculating the most beneficial buy or comparing discounts between two products, calculating the consumption of the petrol in a car trip, or using a map and its scale to calculate the distance between two targets require skills to reason proportionally. Traditional symbolic representations or algorithms linked to proportional reasoning are not familiar for Finnish fifth graders and was therefore chosen as a domain to assess students' intuitive problem-solving skills and strategies in non-routine problems.

Proportional reasoning is often described as a cornerstone to higher mathematical and scientific thinking and cognitive development (e.g. Lesh et al., 1988; Lamon, 2007; 2012). Understanding proportionality requires reasoning with ratios. In textbooks and mathematics dictionaries the word proportion is often defined as an equivalence of ratios or statement of equal ratios or fractions, written as follows:

$$\frac{a}{b} = \frac{c}{d} \text{ or } a : b = c : d.$$

Proportional reasoning requires skills to convey the same relationship for example in producing or comparing ratios or finding a missing value. Abilities to reason proportionally are a marker of a move towards more developed forms of reasoning and form a foundation for example for algebra. Previous research indicates that students are capable of solving proportional word problems already during their early years of primary school (e.g. Tourniaire, 1986; Van Dooren et al., 2005; Vanluydt et al., 2019).

Understanding ratio and proportion requires the ability to reason with multiplicative relationships and distinguish them from relationships, which are additive in nature (Van Dooren et al., 2010; Son, 2013). In an additive approach, the student operates with an invariant difference between two values, whereas a multiplicative approach requires an understanding of an invariant ratio between two values (Van Dooren et al., 2010). Even if some proportional reasoning tasks can be solved by additive approaches, also in those situations students need to understand the co-varying situation of given values. Building-up or scaling-down by skip-counting until the anticipated value is reached represents one of the strategies, which

often bases in additive reasoning. These types of solution methods can be supported as steps towards multiplicative and proportional strategies.

Reasoning is an integral part of mathematical problem-solving and skills reach beyond solving routine problems. Reasoning requires logical and systematic thinking, being a process, which requires making conclusions on how to achieve certain goals; these conclusions guide problem-solving and decision-making behaviour (Leighton, 2004; Grønmo et al., 2013). Students make notions on patterns and regularities and use that information on making decisions on problem-solving approaches. Reasoning involves skills to make conjectures, logical deductions based on assumptions and rules, and abilities to justify results. (Grønmo et al., 2013, p. 27.) Teachers can help students to develop these skills by presenting mathematical problems linked to unfamiliar contexts and providing opportunities to solve open-ended or multi-step problems (e.g. Grønmo et al., 2013). This has not been typically encouraged in school culture (e.g. Pehkonen et al., 2013). Close-ended textbook examples do not necessarily support students' skills to apply the learned procedures or algorithms outside the school context, and the applications to real-world situations can seem rare to them.

### 3 Teaching experiment: Heuristics for problem-solving

Interest towards improving primary-aged students' mathematical problem-solving skills was based on data, which was collected in Finland and Indonesia in 2014-2015 for Kaitera's doctoral research. A preliminary analysis of the mentioned data indicated that Finnish students had severe difficulties in explaining their thinking in tasks. This led to wondering whether these skills could be developed by implementing a teaching approach, which provided tools for solving a wide variety of out-of-the-textbook problems. The teaching experiment was carried out during the following academic year in a class of fifth graders. The learning environment was designed to support the development of students' mathematical problem-solving skills. The quasi-experimental design was conducted in real-world learning settings, attempting to discover aspects that could be useful for example for teachers aiming to develop mathematics teaching practices.

Teaching heuristics for mathematical problem solving is often linked to working with students with challenges in learning mathematics (e.g. Gallagher Landi, 2001; Fuchs & Fuchs, 2003; Swanson et al., 2013). General heuristics are not associated directly to certain kinds of mathematical problems and therefore can facilitate integrating the given information with steps for action (Swanson et al., 2013, p. 170).

This report suggests that any student would benefit from getting familiar with a range of generalisable problem-solving approaches instead of just learning a variety of algorithms fit for certain types of mathematical problems.

### 3.1 Participants and background for the research

Research was carried out in a large urban school in Northern Finland with a class of 25 fifth graders (12 boys and 13 girls). In the beginning of academic year 2015-2016, students' skills and strategies were mapped by a pre-test with proportional reasoning problems. At that time, students' mean age was 11 years and 2 months (range from 10 years and 9 months to 11 years and 7 months). During the autumn semester, the class got familiar with a range of generalisable heuristics, which were used in solving a variety of mathematical problems. Participating class followed the guidelines of mathematics education outlined in the Finnish National Core Curriculum. Students had attended five years of elementary school, but not received any formal instructions in solving proportional reasoning tasks, which were the main domain for assessing the development of mathematical problem-solving skills in this research. The class-teacher had a degree as a Master of Education and had been teaching for 10 years in primary and secondary schools. She was working on her Doctoral research on mathematical problem-solving, and the study described in this report was carried out of an interest towards developing students' problem-solving skills.

At the beginning of the fifth school year, it appeared that many students seemed to struggle in mathematical tasks and especially in explaining their problem-solving processes in written form. Students were introduced to concrete tools called Problem-solving Keys, which were modified from Strategy Keys based on work by Herold-Blasius (2021). The aim was to provide students with a visual reminder of heuristics for mathematical problem-solving tasks. Similar heuristics were outlined also in the Singaporean Mathematics Syllabus 2013 and used as a reference when classifying and modifying the Keys for teaching purposes in Finland.

Fifth graders had three mathematics lessons every week. Mathematics textbooks were used, but in addition to those, during the autumn semester the class spent on a weekly basis on average one mathematics lesson on working with mathematical tasks in a practical context and learning a variety of general heuristics for problem-solving. Out-of-the-textbook problems were solved during the spring semester, too, but learning heuristics was not the focus anymore. Post-test data was collected at the end of the fifth grade in 2016 by using the same test than in the beginning of the school

year. At that time, the mean age of students was 12 years (range from 11 years and 7 months to 12 years and 5 months).

### 3.2 Framework for practicing mathematical problem-solving

The central idea of study in a real-life context was to teach mathematics and general heuristics through solving a variety of out-of-the-textbook problems. Mathematical problems were sourced for example from everyday situations, children's literacy, and local, national and international news. In addition to that, students created word problems for their peers and learned to solve them in various ways. Problems were often integrated into other subjects, such as Environmental Studies and other Science themes, Finnish as a mother tongue and Arts and Crafts.

Exploration of mathematical problems followed a framework with different phases of problem-solving (Stein et al., 2008; OECD, 2014, p. 31): first, the task was presented by the teacher to the students (*a launch phase*), then students worked on problems either in small groups or individually (*an exploration phase, planning and executing*) and finally the outcomes were shared and discussed (*a summarising and reflecting phase*). In practice, the process was not a linear, step-by-step progressing path, but rather a flexible model for moving between different phases. Quite often discussing and sharing the ideas led to returning to the exploration phase and assessing the problem-solving approaches from new perspectives. Polya's (1945/1973) four step model was followed especially during the exploration and summarising phases. Problems were solved in collaborative settings always when it was possible: this enabled discussion and made the importance of justifying thinking more visible.

Heuristics or general techniques for solving mathematical problems were introduced to students by using a visual tool called Problem-solving Keys, which are based on for example Polya's (1945/1973) and Bruder and Collet's (2011) heuristics, and the same ideas were outlined in Singaporean Mathematics Curriculum 2013. These heuristics were modified into a concrete tool by Herold-Blasius and Rott (2016) and named as Strategy Keys. They describe these tools as "door openers" for a problem-solving process and reminders of general heuristics that students have learned (Herold-Blasius & Rott, 2016; Herold-Blasius, 2021).

Keys were modified for teaching experiment purposes, translated in Finnish, and renamed as Problem-solving Keys. Keys that were used in this study were chosen based on their generalisability, transferability and fit for the mathematics curriculum

for this age group. The guidance progressed by introducing one or two keys (heuristics) at the time, linking them in a variety of out-of-the-textbook problems. As the new heuristic was introduced and practised, the key linked to that particular heuristic was added to a student's personal "Problem-solving key chain". Each key was linked to a mathematical problem, which was often open-ended, or at least had multiple different solution paths to choose from. The problem was chosen so that the heuristic in that Key worked well in solving a particular problem: for example, Gravett's The Rabbit Problem (2009) was based in Fibonacci's approach and used when practising the problem solving by using a table. Literature often offers an excellent context to bring abstract and complicated concepts closer to real-world situations. The following Figure 1 shortly illustrates the keys which were chosen as a focus area in this study, and some prompts, which were presented in guiding the learning processes.

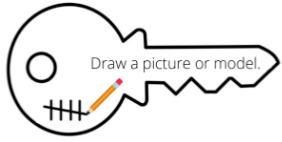




	<p><b>Draw a picture or model</b> Create a visual representation (drawing, icon, simplified picture etc.) of the situation. Use the picture or model as an aid in solving the problem.</p>						
 <table border="1" data-bbox="443 1149 518 1200"> <tr><td>cats</td><td>6</td></tr> <tr><td>dogs</td><td>4</td></tr> <tr><td>horses</td><td>5</td></tr> </table>	cats	6	dogs	4	horses	5	<p><b>Make a table</b> Arrange data by using a table. Fill in the data which is already known. Can you recognise conditions, which are affecting given data? How do these conditions help you in finding out the missing information?</p>
cats	6						
dogs	4						
horses	5						
	<p><b>Look for a pattern or rule</b> Arrange data by using e.g. tables or diagrams. Can you recognise rules or patterns? Can you make predictions and test them? This heuristic can often be linked to a previous Key, Make a table.</p>						
	<p><b>Draw a diagram</b> Create a suitable diagram based on the given information. Can you make assumptions based on that? How can you interpret the diagram? Are there any outliers and if yes, why?</p>						
	<p><b>Use different colours</b> Find and mark the important information by using different colours. Will this help you to progress in the process and understand the question better?</p>						

Figure 1. Examples of Problem-solving Keys and prompts presented to students.

In addition to the keys described in Figure 1, students had three additional keys, which were called "When I'm stuck" -keys:



- Read the task again,
- Guess and check,
- Solve part of the problem.

These ideas often enabled either using other heuristics or continuing with other steps in the process. Key called “Solve part of the problem” turned out to be well used. Breaking the problem down into more approachable steps and solving even a small part of the problem opened new insights on how to proceed in tasks. The notion is not new: Duncker (1945, p. 8) linked “reformulation of the original problem” as one of the important characteristics in the problem-solving process and referred to this reformulation as a step or phase on a path towards solution. As Kilpatrick (2016, p. 45) points out, it might be easier to solve the problem if it is broken into smaller pieces or modified into another form.

It is important that teachers value attempts for intuitive problem-solving methods and will be able to guide the student forward. Children often use everyday logic and apply that also to mathematical problems. They can be invited to justify their thinking and invent proofs for their ideas. Later students should learn about mathematical proof and formalities. They need to recognise that there is a difference between a guess, a conjecture, and a proven assertion. It is important to encourage students to wonder why things are as they are and guide them in providing a logical chain of reasons as the explanation. (Goldenberg et al., 2003, p. 24.) An educated guess differs from a random guess by its metacognitive aspects. True mathematical problem solving is challenging, but at the same time rewarding for both students and the teacher, as Schoenfeld (1992, p. 354) points out.

Students benefit from having opportunities to explain their thinking not only by using mathematical language, but also pictorial and natural language: possibilities to draw and write during the problem-solving process may strengthen the understanding of mathematical concepts and contribute to mathematical thinking skills (Joutsenlahti & Kulju, 2017). Open-ended problems or planted error tasks are excellent domains for developing students’ skills in negotiating and articulating their mathematical ideas to others. According to D’Ambrosio and Prevost (2008, p. 276) “all contributions should be valued and respected“. By assessing students’ solution methods, also the self-generated ones, teachers can correct the ones which are mathematically acceptable, or guide students forward in partially constructed explanations. Classroom discussions provide crucial information on students’ understanding on topic and problem-solving processes. Effective teaching includes



listening to students' ideas and explanations and using that information as a guide in making decisions on instruction (e.g. Lester, 2013; Ivars et al. 2020; Shaughnessy et al., 2021). These views were at the centre point of the study, because a variety of out-of-the-textbook problems enabled interesting mathematical discussions in the classroom. Conversations were emphasized as important steps in learning problem-solving. Students were advised and expected to show their thinking in tasks by writing down the calculations or drawing the stages in solving the problem in a mathematically understandable way. That can be surprisingly difficult even for the 10-12-year-old students, who have already attended several mathematics lessons per week for multiple years.

## 4 Mapping the problem-solving skills

### 4.1 Data collection instruments

Students' performance was assessed by individually completed paper-and-pencil tests, which were taken in the beginning and in the end of fifth grade. Tasks included different types of proportional reasoning problems and are presented in more detail in Table 1 and Table 2. Students had a 45-minute lesson to complete the pre- and post-tests, but most of them used 20-30 minutes for tasks.

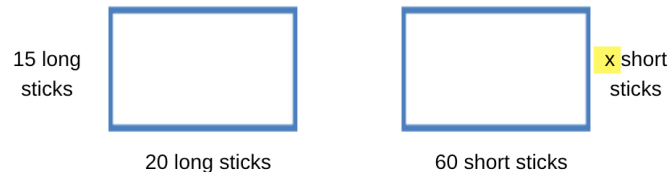
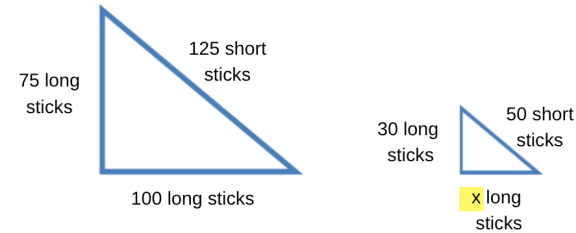
Multiple-choice questions 1-5 and 8 represented typical comparison problems, in which students needed to determine the relationship(s) of two or more ratios, for example by judging whether one ratio is greater or less than the other one(s) or are they equal. In task three with mixtures two of the given ratios were similar. The rationale for having two equivalent ratios in the task was to map whether students were favouring one of the choices over the other, in this case whether they took the first choice, 2:4 or rather chose 1:2, which is used in several everyday contexts.

**Table 1.** Proportional reasoning tasks with ratios (multiple-choice questions)

Item	Description	Context and source
1. Lemon tea	Mum is making lemon tea. She mixes tea and sugar in a jug.  Which one tastes the most sweet? Choose. 1 glass of tea and 1 spoon of sugar 4 glasses of tea and 4 spoons of sugar 1 glass of tea and 3 spoons of sugar	Comparing ratios in mixtures (qualitative comparison), adapted from Noelting (1980) and Kaput and West (1994)
2. Lemon tea	Which one tastes the most sweet? Choose. 1 glass of tea and 2 spoons of sugar 2 glasses of tea and 2 spoons of sugar 2 glasses of tea and 1 spoon of sugar	
3. Lemon tea	Which one tastes the most sweet? Choose. 2 glasses of tea and 3 spoons of sugar 1 glass of tea and 2 spoons of sugar 2 glasses of tea and 3 spoons of sugar	
4. Lemon tea	Which one tastes the most sweet? Choose. 2 glasses of tea and 3 spoons of sugar 1 glass of tea and 2 spoons of sugar 1 glass of tea and 3 spoons of sugar	
5. Lemon tea	Which one tastes the most sweet? Choose. 6 glasses of tea and 3 spoons of sugar 5 glass of tea and 2 spoons of sugar 5 glasses of tea and 3 spoons of sugar	
8. Paint-mixture	Green paint is made by mixing two buckets of blue paint and three buckets of yellow paint. The painter needs to get more paint. How many buckets of blue and yellow paint does he need to get the exactly same shade of green? Choose one option. 3 buckets of blue and 4 buckets of yellow paint 4 buckets of blue and 6 buckets of yellow paint 5 buckets of blue and 6 buckets of yellow paint 6 buckets of blue and 8 buckets of yellow paint	Comparing ratios (quantitative comparison), similarity, adapted from Tourniaire (1986)

Tasks 6A, 6B, 7 and 9 required proportional reasoning with ratios, inverse proportionality or similarity of mixtures. Students were explicitly asked to record their thinking in these tasks and explain their problem-solving processes by mathematical, pictorial and/or natural language in written form.

**Table 2.** Tasks used in assessing strategies: Proportional reasoning tasks with ratios, inverse proportionality or similarity of mixtures

Item	Description	Context and source
6A. Rectangles	Students are building geometric shapes. They make two similar rectangles and triangles by using short and long sticks.  How many sticks do they need in x?	Determining a missing value with continuous ratio-preserving, geometric similarity, idea adapted from Mr. Tall and Mr. Small tasks by Karplus et al. (1974), Lamon (1993) and research by Son (2013)
		
6B. Triangles	How many sticks do they need in x?	
		
7. Painters	Six painters paint a house in three days. If they all work at the same speed, how many painters would be needed to paint the same house in one day?	Inverse proportionality, item was created for this research
9. Paint-mixture	Orange paint is made by mixing four buckets of yellow paint and one bucket of red paint. To get exactly the same shade of orange, how many buckets of red would the painter need to mix to six buckets of yellow?	Comparing ratios (quantitative comparison), similarity, adapted from Tourniaire (1986)

Timing of pre-test was before students were introduced Problem-solving Keys, and therefore they were not used while solving the items. Post-test was in the end of the school year and students were allowed to use their “key chain”, if they wished, in a similar way they could during the ordinary mathematical tests as well. None of the students felt that they needed the Problem-solving Keys at the post-test in June. The aim of this study was not to map the role or usage of these tools for heuristics but would be another interesting viewpoint for the future research (see Herold-Blasius, 2021). Problem-solving Keys were on a very important role when teaching different

ways to approach a wide variety of non-routine mathematical problems especially during the autumn semester.

## 4.2 Data analysis

First, students' overall test performance in tasks 1-9 was assessed by awarding points on correct and erroneous answers, but also on intermediate steps towards correct explanation. In multiple choice items 1-5 and 8, students received 1 point for a correct answer and 0 points for an erroneous one. In item 3, students were expected to choose both options A and B to gain 1 point, and 0,5 points were given, if they chose either A or B. Maximum points for multiple choice questions were 6. In items 6A, 6B, 7 and 9 students were expected to explain their thinking, and their answers were serving as a base for building a framework for correct and erroneous strategies from intuitive to more sophisticated ones. Maximum points for these items were 2 points for each. The in-between marks were the following:

- 0 points: erroneous explanation and/or answer, or no answer provided
- 0,5 points: some explanation towards correct answer provided, answer incorrect
- 1 point: no explanation provided, answer correct
- 1,5 points: some explanation provided, answer correct
- 2 points: correct explanation provided, answer correct.

With this grading, it was possible to gain a maximum of 8 points in items 6A, 6B, 7 and 9. This approach was close to rating used in school mathematics tests for this age group, and took also the partially correct answers into account. Numerical scores were used as indicators of overall performance and possible development between pre- and post-tests. Maximum points for the whole test were 14.

Exploring and mapping the strategies that students used in task began by dismantling the data (students' responses in items 6A, 6B, 7 and 9). This was done on a detailed level by creating codes based on how students justified their thinking and explained it by using numbers, drawings or written explanations. Coding was concluded with Grounded Theory methods, which provide systematic, yet flexible guidelines for collecting and analysing data (Charmaz, 2014; Birks & Mills, 2015; Chun Tie et al., 2019). Written explanations were worked through in three phases of analysis (Charmaz, 2014), and the framework for coding was created by classifying similar responses to sub-categories (focused coding phase) and core categories

(theoretical coding phase). This scheme was used as the analytical tool to assess the strategies that students used in solving tasks and on the other hand, as an indicator on whether the teaching approach provoked a shift from intuitive to more sophisticated heuristics linked to proportional reasoning.

Table 3 illustrates students' correct answers in Task 6A, and how they were grouped as sub-categories during the focused coding phase.

**Table 3.** Dismantling the data during the initial coding phase and sub-categories in focused coding phase: example from correct approaches in Task 6A

Initial coding phase: observable behaviour	Focused coding phase: sub-category
Student understands that the long side on the second rectangle is "three times longer" than the corresponding side on the first rectangle. Demonstrates thinking by addition: $20+20+20=60$ and $15+15+15=45$ , but cannot clearly explain where "three times" comes from.	Demonstration of relative thinking between given quantities but failing to provide mathematically understandable explanations.
Student understands that 20 long sticks = 60 short sticks by comparing corresponding parts but cannot explain how he/she gets $x=45$ .	
Student calculates the ratio between the sides of the first rectangle and applies the same logic to another picture.	Demonstration of relative thinking between quantities e.g. by using ratio as a unit in calculations, but not necessarily able to create generalisable formulas.
Student calculates that on the first rectangle the vertical side is $\frac{3}{4}$ of the horizontal side and applies the logic to the second rectangle to determine $x$ (for example by deducting $\frac{1}{4}$ from 60).	
Student understands that long sticks are three times longer than short sticks, and is able to utilize the knowledge to solve missing value $x$ .	
Student works with both rectangles simultaneously by using the ratio 3:4 to solve the missing value (ability to form generalisable calculations).	Use of formal operations based on ratio or use of a certain algorithm, such as cross-multiplication or "rule of three".
Student uses a formula, such as a cross-multiplication algorithm, "rule of three" or equivalent to solve the task.	

Consistency for the coding scheme was ensured by comparing the original data in several phases of the coding by student to another student, student's answer to anticipated strategy and strategy by strategy. This involved repeated visits to original answers to ensure that they were understood and interpreted correctly. Final scheme for coding can be found in Appendix 1.

## 5 Results

### 5.1 Performance in tasks

Tasks 1-5 and 8 were multiple choice mixture tasks, and even though the student's choice of option could give some indication on solution strategy as well, this report focuses on analysis and classification of solution approaches in tasks 6A, 6B, 7 and 9.

Students' performance in tests provided insights on whether the teaching approach, which focused on mathematical problem-solving, improved students' general skills in solving also proportional reasoning tasks. In the beginning of fifth grade, the mean for total score in the proportional reasoning test was 6,1 points (SD 2,5 p.). Boys (N=12) performed better than girls, their mean being 6,5 points (SD 2,1 p., minimum 3,5 p. and maximum 11 p.), whereas girls (N=12, one being absent) had a mean of 5,5 points (SD 2,9 p., minimum 0,5 p. and maximum 12 p.).

After getting familiar with a variety of different heuristics (but not explicitly algorithms) for solving mathematical problems, the post-test in June indicated positive results: the mean score of students had risen to 8,9 points (SD 3,6 p.). It was interesting to notice that this time girls performed better than boys. Female students' mean had risen from pre-tests' 5,5 points to 9,3 points (SD 3,3 p., min. 5 p. and max. 13,5 p.). Male students also improved their performance: in the pre-test they had a mean of 6,5 points and in the post-test 8,4 points (SD 4 p., min. 0,5 p., max. 13,5 p.). Development of total points is illustrated in [Figure 2](#).

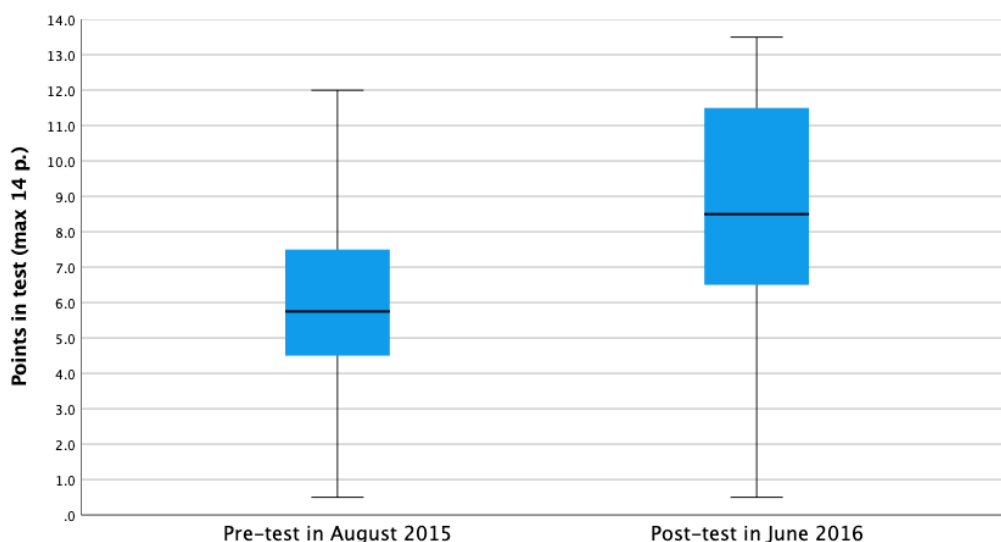


Figure 2. Total points in pre- and post-tests.



The following [Figure 3](#) visualises individual students' performance. Blue marks indicate an individual's total points in the beginning of the fifth grade, whereas orange marks are for post-test points in the end of the school year. Development of skills was visible especially among those students, who in the pre-test scored below the average points, but it seems that the intervention had a positive influence on skills of almost all students<sup>1</sup>.

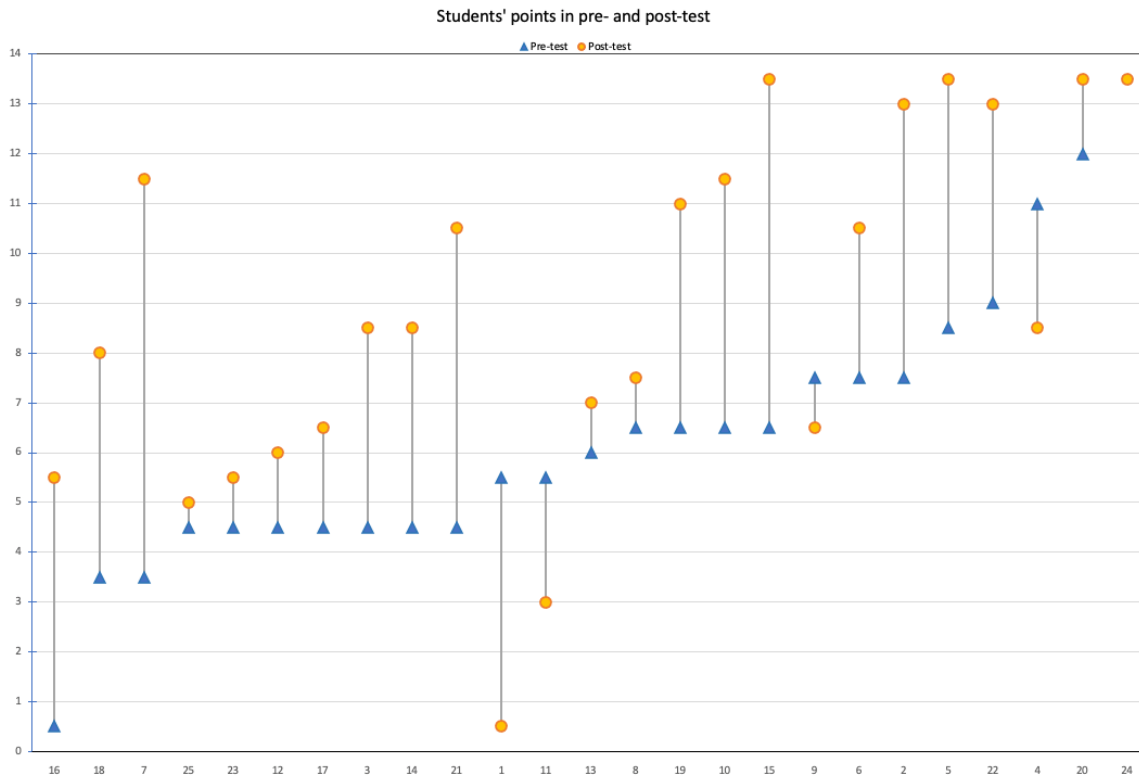


Figure 3. Development of total points by individual students in problem-solving pre-test in August and post-test in July.

Difficulty of tasks is often linked to the number structure and numerical complexity. For example, mathematical problems with small, integer ratios are easier than tasks with non-integer ratios (e.g. Tourniaire, 1986). For assessing the difficulty of items, students' answers were combined with a larger set of data from Finnish fifth graders', which completed the same test. Difficulty of items was done by assessing frequencies of correct and erroneous answers by 95 students. Items in the test sheets

<sup>1</sup> Student number 24 was absent during pre-test tasks 7-9 and the total points are not calculated. In the post-test, student 1 did not answer any of the questions 6-9, which affected the final score. Student 11 left several tasks unanswered, or it was not possible to determine the answer.

were designed to get gradually more challenging, but it seems that the difficulty of items in this test did not match students' skills and therefore the interpretations on students' performance need to be addressed with reservations. Tasks 1-5 were too easy for fifth graders, and on the other hand the success rate in tasks 6A, 6B and 9 was 20-28%. With this setting, the difficulty of task 7 was fairly ideal (success rate 58%) and task 8 was almost too difficult. If the results of the post-test would be considered, too difficult items would appear to be more ideal also in tasks, which required skills to explain reasoning.

Results indicate that there was no significant improvement in how students performed in multiple-choice mixture tasks in pre- and post-tests. High success rates suggest that tasks 1-5 were easy for fifth graders at the first place. Development of skills is visible in more difficult tasks 6, 7, 8 and 9, which are discussed in more detail in Chapter 5.2. These results indicate that the teaching approach with a focus on problem-solving may have had a positive influence on students' abilities to solve tasks, which require proportional reasoning skills.

Mathematical problems were usually not presented in a written form in a similar way as typical word problems in mathematics books. Students did not get any extra training in solving word problems and therefore the development of skills cannot be explained by them getting more fluent in solving mathematical problems presented in written form. A table describing students' ability to solve tasks correctly can be accessed in Appendix 2 and will be discussed task by task in the next sub-chapter.

## 5.2 Strategies in tasks

One of the aims of the study was to find out if teaching approach, which offered tools for heuristics, improved students' skills in explaining their thinking in mathematical tasks. In addition to getting familiar with problem-solving techniques, the practical aim for the intervention was to build up students' mathematical self-confidence so that they would become active in describing their problem-solving processes. Questions 6A, 6B, 7 and 9 were assessed as indicators, if students were able to give an understandable explanation on how they processed the task. Informal techniques and strategies provided an insight on how students understood problem-solving concepts and were able to progress even in an unfamiliar type of a problem. Explanations and heuristics were also assessed to see if there were differences in students' use of correct and erroneous strategies between the pre-test and the post-test.

In the beginning of fifth grade, students had major difficulties in describing their problem-solving path and often left the explanation completely out. Teaching approach, which encouraged students to describe their thinking even with partially complete explanations and solving problems one step after another, seemed to have a positive impact on their performance during the later phases of the academic year.

### 5.2.1 Task 6A: Rectangles

Tasks 6A and 6B represented typical proportional reasoning problems with a missing value. According to Karplus et al. (1983, p. 21), these types of problems involve “reasoning in a system of two variables between which there exists a linear functional relationship”. To maintain proportional values, students carry out parallel transformations within or between variables (Son, 2013). The relation between quantities is invariant, whereas the quantities in the problem co-vary.

In task 6A, a correct approach required the ability to compare corresponding parts between two rectangles. 17 students (68%) provided an answer to the question 6A, and six students (24%) were able to solve the task correctly. 10 students out of 17 were able to explain their solution process, whether the answer was erroneous or correct. Almost a quarter of all students (N=6) were skilled enough to explain their thinking with the correct approach. Seven students provided an answer but did not explain how they ended up in that. Eight students (32%) did not answer the question at all.

It appears that the problem-based teaching approach had a positive impact on students’ skills: in the post-test 92% of students (N=23) answered the question and 68% (N=17) were able to provide a correct answer. 20 students out of 23, who answered the question, were able to explain their thinking in written form. Almost a half (N=12) of all students in the post-test approached the task with the correct strategy. Only three students answered the question but did not explain their thinking and two students (8%) did not answer the question at all.

In pre-test 16% (N=4) were able to implement a correct ratio or unit factor approach in task 6A, demonstrating relative thinking between quantities in solving the unknown quantity. In post-test the number of students using a correct strategy had more than doubled, being 40% (N=10). Even though in many cases an explanation for the solution process did not include all the mathematically correct steps, students were demonstrating the understanding of long sticks being three times longer than the short sticks.

During the pre-test, the most common erroneous strategy was additive reasoning (16%,  $N=4$ ). It was typical to focus on dimensions within one rectangle, for example reasoning that because the difference between the sides of the first figure was five ( $20-15=5$ ), the same difference applies for the second figure ( $60-5=55$ ). In some cases, students calculated the perimeter of the first rectangle and tried to apply or modify the logic to find the missing value in the second rectangle. In both examples students failed to understand the relational nature of the task: if 20 long sticks equal the length of 60 short sticks, the same ratio should be maintained with 15 long sticks and  $x$  short sticks. According to the previous research, students often rely on additive strategies also in multiplicative situations (e.g. Tourniaire & Pulos, 1985; Nunes & Bryant, 1996; Van Dooren et al., 2010; Son, 2013). Still, it is not clear how students choose their preferences between additive and multiplicative relations (Vanluydt et al., 2019). Both approaches can be characterised as intuitive in nature, yet it is difficult to verbally describe reasoning; the given explanations are not necessarily in line with students' actual solution processes (Degrande et al., 2020).

Distinguishing multiplicative missing value problems from additive ones is challenging for students. Additive thinking is emphasized during the first years of school and the transition towards multiplicative ideas is not always straightforward. On the other hand, additive reasoning could support the development of multiplicative reasoning. Yet, the shift from additive to multiplicative thinking requires a qualitative change in thinking (e.g. Nunes & Bryant, 1996). In the beginning of the fifth grade, only one student approached the problem via multiplicative reasoning but ended up in an erroneous end-result. After the intervention, one fifth ( $N=5$ ) of students turned into this approach. Even though these solution attempts were erroneous, they could be interpreted as a shift towards understanding the relative nature of the task. A more detailed description on the range of strategies that students used can be accessed in Appendix 3.

Development of solution approaches and possible shifts between the strategies was visualised as individual students' performance in tasks. In Figure 4, explanation categories are presented in an order, which suggests a hierarchy from erroneous and intuitive ones to more sophisticated and generalisable strategies. Light green area marks correct approaches. Opaque fill-ins in pre- and post-test markers indicate that the student was able to solve the task correctly.

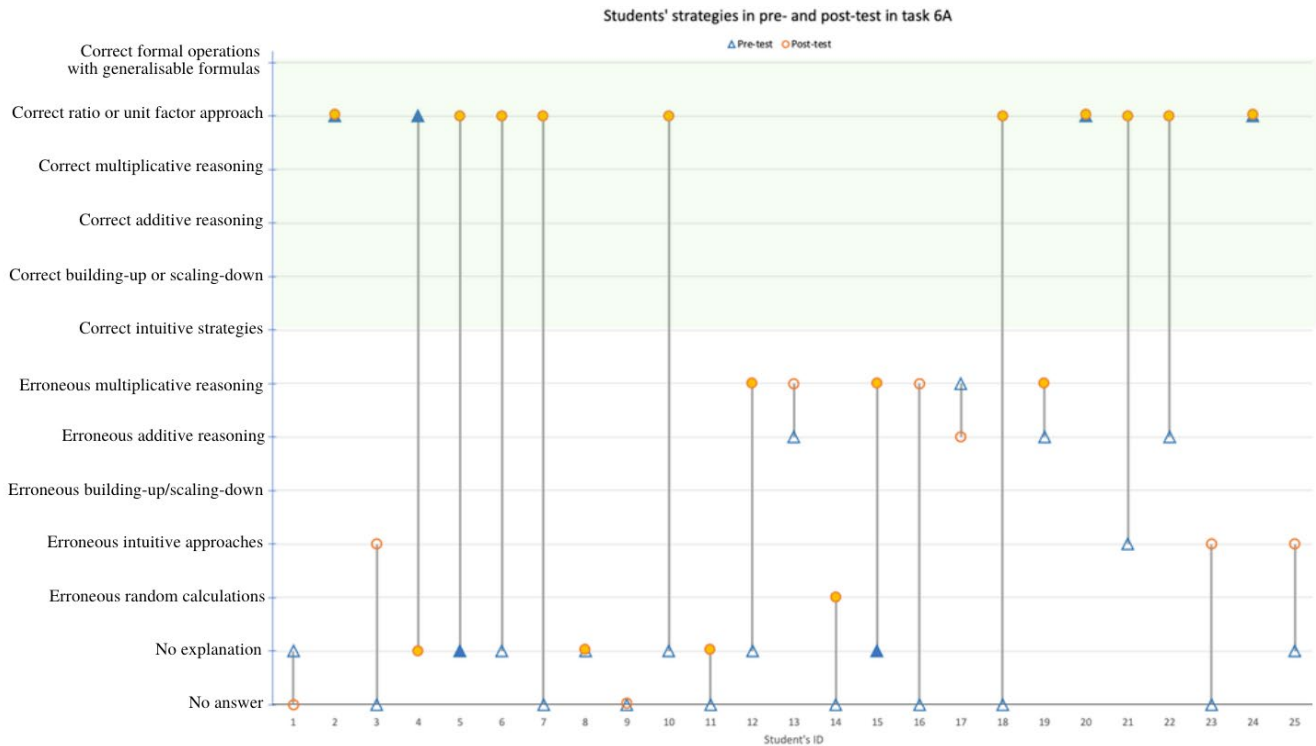


Figure 4. Individual students' strategies in pre- and post-tests in task 6A.

Correct solution approaches were rare in the pre-test, even though the task was relatively easy. In the end of the fifth school year the frequency for correct strategies had increased and students were able to approach the task by correct ratio or unit factor approach.

### 5.2.2 Task 6B: Triangles

Task 6B was more difficult than 6A. It would have been possible to solve the task only by focusing on dimensions on one triangle and using Pythagorean theorem, but that is a topic for Finnish secondary school curriculum and therefore not expected that any of the students would use that algorithm. In the pre-test 16 students (64%) answered task 6B and only two (8%) of them solved the task correctly. Seven students of 16 explained their thinking process in writing, but only one of them was able to choose a correct strategy. Nine students gave an answer, but no explanation. Nine students (36%) did not answer the question 6B in pre-test.

Before the intervention, students had difficulties in explaining their thinking, 72% of students (N=18) either leaving the explanation out (N=9) or not answering the question at all (N=9). By the end of the school year, the number of empty explanation spaces (36%, N=9) had decreased to half, even though the task was challenging. In the

post-test 76% of students (N=19) answered the question and less than a quarter did not (N=6). 48% (N=12) of students were able to provide a correct answer. The majority, 16 students out of 19, tried to explain their thinking in a written form. Three students answered the question but did not provide any insights on the solution process. In the pre-test only one student was able to choose a correct strategy, but in the post-test the number increased to seven students (37% of 19 students answering this question). This was quite an interesting finding, because students had not encountered any similar mathematical problems during the academic year.

In the pre-test, only one student was able to explain thinking by demonstrating mathematically correct reasoning. During the intervention the variety of correct solution strategies increased. Students came to conclusions by additive reasoning or more sophisticated multiplicative reasoning, and there were also some examples of abilities to create correct, generalisable formulas to solve these types of problems. None of the students solved the task by using ratio or unit factor.

Students often relied on erroneous intuitive strategies, such as trying to solve the problem by random calculations on given numbers or basing the problem-solving process on visual observations on given pictures, and not mathematically valid concepts. The range of observable strategies in this task can be accessed in Appendix 3. Figure 5 illustrates the changes in used strategies that individual students had between from the pre-test and to the post-test.

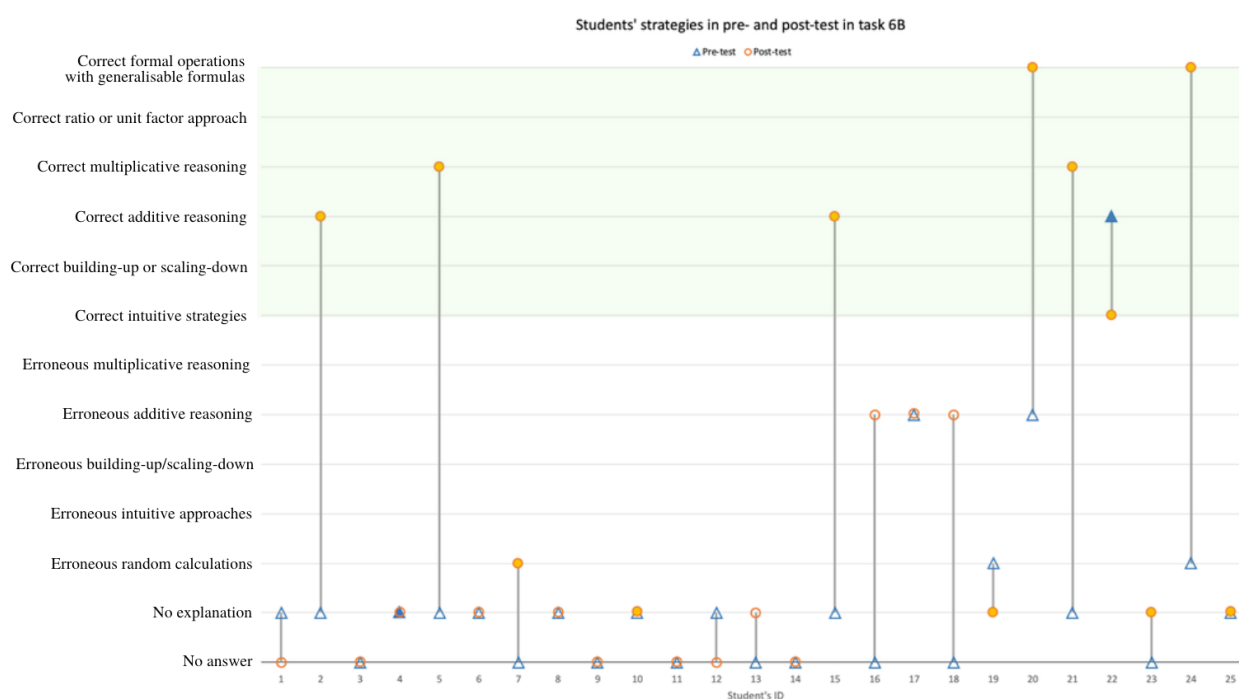


Figure 5. Individual students' strategies in pre- and post-tests in task 6B.



As one can see from [Figure 5](#), task 6B was more difficult than 6A for this student group. After the teaching experiment, successful students chose usually correct additive or multiplicative reasoning, but many students left the explanation out still during the post-test.

### 5.2.3 Task 7: Painters

Seventh task was based on inverse proportionality. Painting a house involved a situation, in which the time spent on painting was reduced from three days to one day, and students were calculating the number of people needed in painting work. In this task, it was crucial to understand that it would take three times as many painters to complete the work in  $\frac{1}{3}$  of the time. The analysis of students' responses raised a question, whether many of them solved the task correctly without really understanding the concept. Due to the numerical structure in this task, it was possible to end up in a correct answer of 18 painters by simply multiplying the word problems' given numbers, six and three.

67% (N=16) of students in the pre-test solved the problem correctly, and in the post-test the frequency had increased to 80% (N=20). Only one student in both tests did not answer the question at all. In the pre-test 70% of students (N=16) who answered the questions also explained their thinking, but only one of them was able to choose a correct strategy. In the post-test 24 out of 25 students gave an explanation on their solution process, and at that point 83% (N=20) of them used the correct approach. In the post-test none of them left the explanation slot empty. High success rates in all student groups are possibly linked also to the possible bias caused by the number structure. Majority of students based their explanation on this particular task simply stating  $3 \times 6 = 18$  but did not provide any additional information on how they were thinking, or where the numbers came from. Only a few of the participants with correct answers were able to express that they understood the concept instead of performing a random calculation. They, for example, reasoned the number of painters by building up or scaling down with the figures (e.g., 6 painters=3 days, 12 painters=2 days, 18 painters=1 day) or used the addition or multiplication, but were rarely able to justify, why they chose certain procedures. Even though multiplicative reasoning was the most common correct strategy, it is difficult to assess whether the concept of inverse proportionality was really understood. Range of strategies in task 7 can be accessed in Appendix 4 and development of strategies between pre- and post-tests in [Figure 6](#).

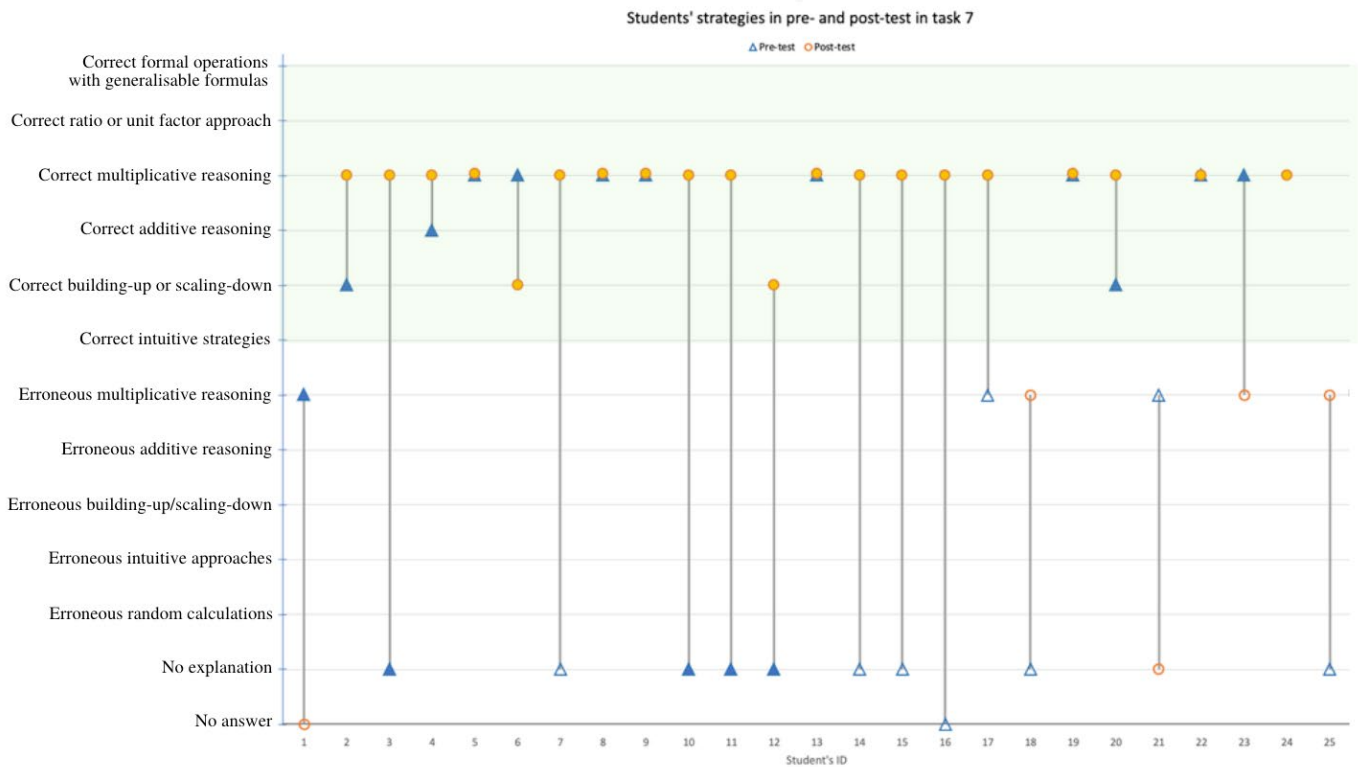


Figure 6. Individual students' strategies in pre- and post-tests in task 7.

It is likely that the wording and the number structure of this task also guided the choice of erroneous approaches: students often relied on erroneous multiplicative reasoning, which was the most common erroneous strategy. To map the real understanding of inverse proportionality, the task could be worded for example by “Two painters paint the house in three days. If they all work at the same speed, how many painters would be needed to paint the same house in two days?” In this case, multiplying two by three would not result in a correct answer.

#### 5.2.4 Task 9: Paint-mixtures

Task 9 was a mixture task, in which students had to maintain the same ratio of paint buckets per mixture to determine the missing value (number of red paint buckets) for the similar mixture. This was a difficult test item for fifth graders, but on the other hand, provided interesting insights on students' development of problem-solving skills.

During the pre-test 29% of students (N=7) did not answer the question at all, whereas the percentage in the comparison group was lower, 14% (N=7). In the post-test, only two students did not answer the question and in both cases, they expressed their unwillingness to engage with the task at all.

17% of students (N=4) had a correct answer in pre-test, but clearly struggled in providing explanations on their reasoning processes: 12 out of 17 students, who answered the question, left the explanation out. Only one student was able to choose the correct strategy in this task, the other four relied on erroneous approaches. Post-test results indicated significant improvement. 52% of students (N=13) ended up with a correct answer, and 10 out of 23 students answering this question also described their thinking with a correct strategy.

When having a closer look on strategies that students used, in the pre-test only one student was able to provide an explanation while solving the problem correctly, turning into a building-up strategy. Development of skills was visible in the post-test: more students were able to not only explain their correct problem-solving process, but also use a more sophisticated strategy by working with the ratio. Even though students did not necessarily have skills to explain thinking with mathematically valid expressions, they became more confident in using different strategies. [Figure 7](#) illustrates the ratio approach, in which student proceeds one step at the time. In this example, the student correctly reasons that because there is one red paint bucket in every four yellow paint buckets, you need to add 1,5 buckets of red to six buckets of yellow.

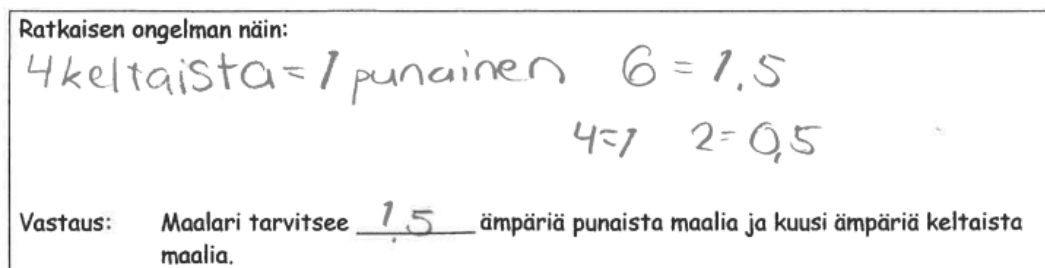


Figure 7. Correct example in item 9 (student 42159 in post-test).

During the pre-test, only one of the students was able to provide a correct explanation for the task, working with building-up strategy. In the end of the school year there were indications on improved skills of explaining thinking also visible: 32% (N=8) utilised either building-up or scaling-down strategy, ratio or unit factor approach (the most common) or even correct formal operations with generalisable formulas. In the post-test, five students (20%) were able to work through the task by expressing that for every two buckets of yellow you need 0,5 buckets of red paint.

54% (N=13) of students gave an erroneous answer in the pre-test. Three students relied on multiplicative reasoning but failed to understand the relative nature of the

task. They multiplied the given amounts of yellow paint,  $6 \times 4 = 24$  or only stated “Calculated by multiplication”, without providing a more detailed explanation.

During the post-test the most common erroneous strategy was additive reasoning. 24% (N=6) of students chose that strategy. They often based their reasoning on the idea that “you need three more yellow than red”, focusing on the difference between the given numbers in the original paint mixture and ignoring the need to maintain the same relationship for the second paint. More detailed frequencies for the strategies visible in task 9 can be accessed in Appendix 4.

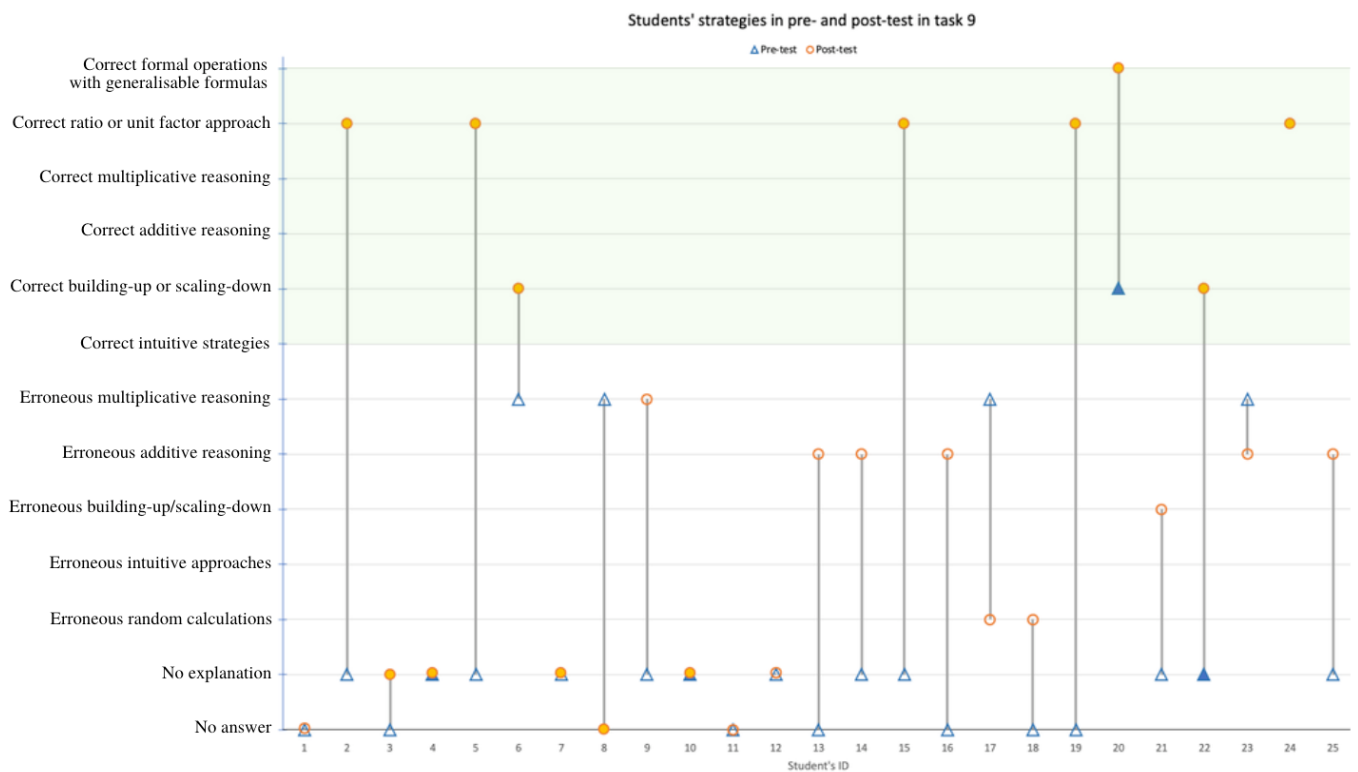


Figure 8. Individual students’ strategies in pre- and post-tests in task 9.

Assessing and classifying students’ strategies was not always straightforward. For example, student could state that multiplication was needed, but on the other hand, relied on additive reasoning when providing an answer: “*Because in the beginning you needed three more yellow buckets than red buckets, so you just need to multiply it*”, providing three as an answer.

### 5.2.5 Students with the lowest and highest points

To have a closer look on possible development of strategies of so-called low- and high-performing students, the performance of three students with lowest points and four students with the highest points in the pre-test were considered. Three low-performing students gained a maximum of 3,5 points in the pre-test and four high-performing students 8,5-12 points (for the overview of students' performance, see [Figure 3](#)).

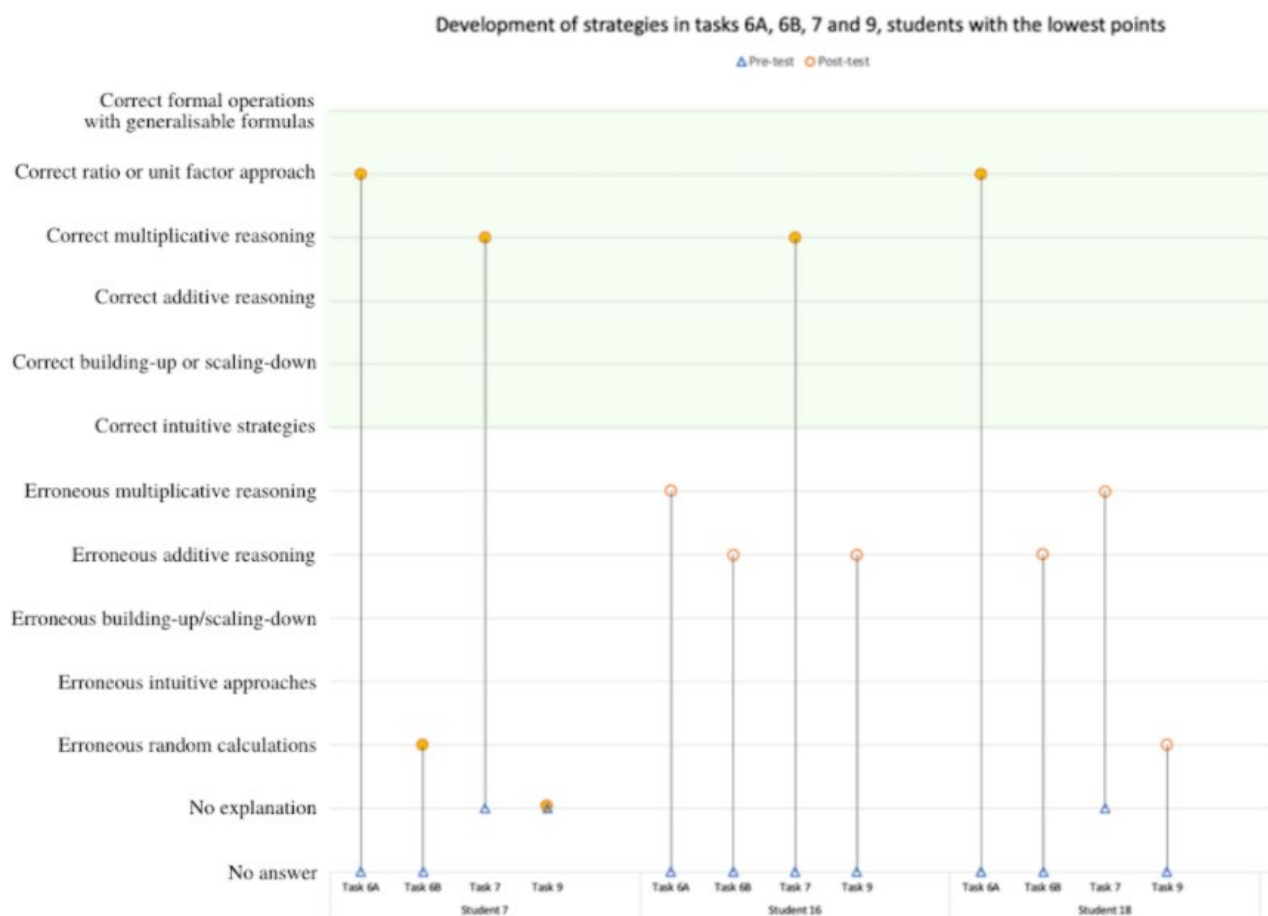


Figure 9. Development of strategies of three students scoring the lowest points in pre-test.

These three students tended to leave the answers completely out in the beginning of the fifth grade. By the end of the fifth grade, frequencies for solving the tasks correctly increased. With some individuals the difference was remarkable: for example, student 7 got correct answers in the post-test, but would still have needed a bit of support in explaining thinking (see [Figure 9](#)). After the problem-based teaching period, students were more willing to engage in attempts to solve mathematical problems, even though the strategies might not have been valid. With a correctly

timed intervention the teacher has a chance to support the shift from erroneous strategies to correct ones.

If lacking the skills to explain thinking with mathematically correct processes, students often started to explore the dimensions between the given values by implementing intuitive methods. Consider the explanation in [Figure 10](#) that student 22 gave in Task 6B: the answer was correct 40 sticks, and in this case, the student seemed to calculate the solution by exploring the given values and their relationships within the first triangle. This student calculated the difference between the hypotenuse and opposite side is multiplied by two to get the adjacent side.

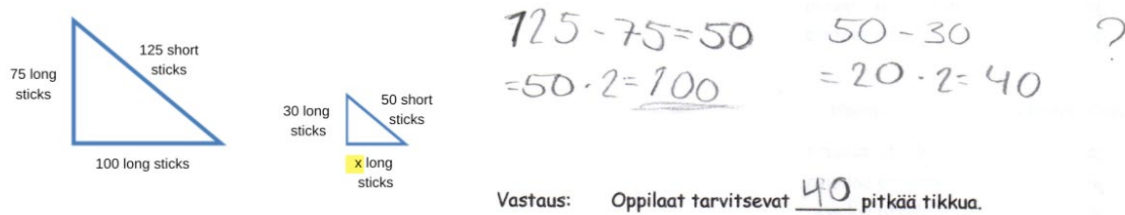


Figure 10. It was not uncommon that the answers were correct, but not necessarily based on generalisable ideas.

In these kinds of examples, which are very common in primary school, students would benefit from opportunities to discuss their ideas with the teacher or with a peer; what is the purpose of short and long sticks in this task, and how should that information guide the solution process? If the strategy works with the given values, can that be generalised to all triangles with a 90-degree angle? How about all types of triangles?

Findings of the study indicated that especially students with lower points benefited from exploring different problems and heuristics to approach them. Students with high points in the pre-test were able to develop their skills in explaining their ideas and on the other hand, to move towards more advanced strategies (see [Figure 11](#)).



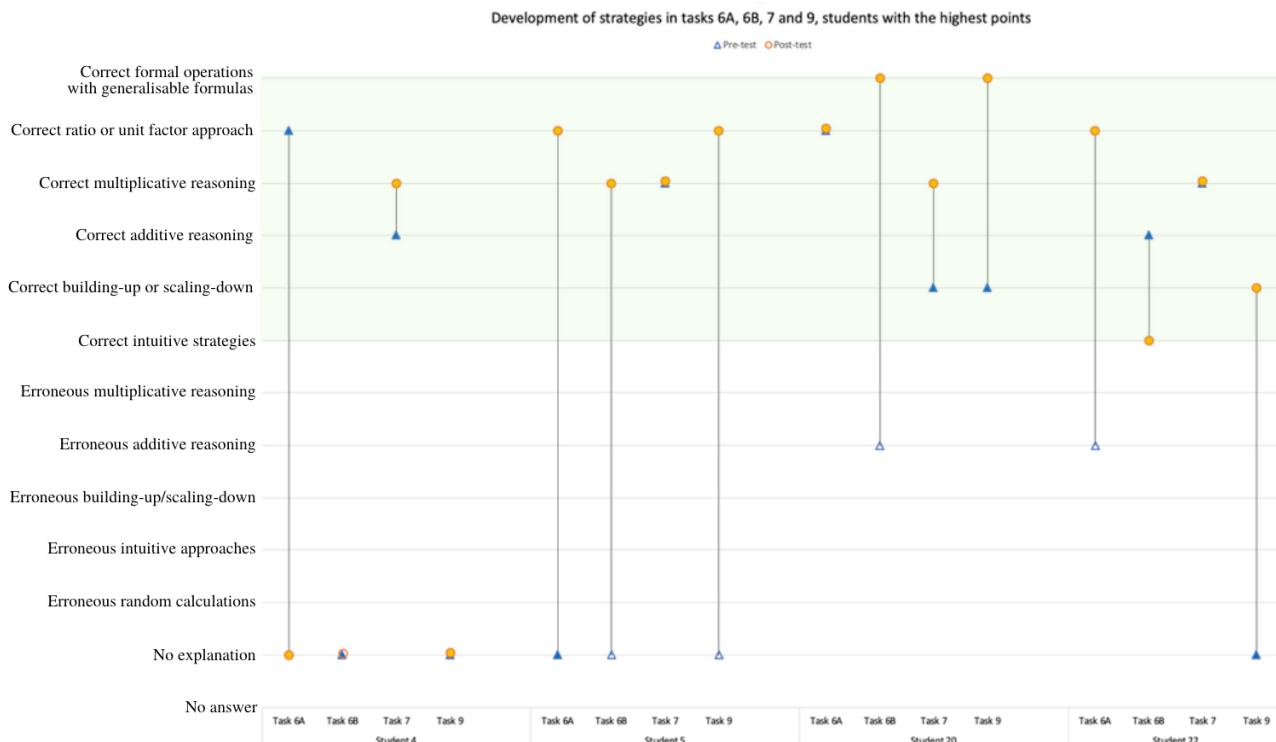


Figure 11. Development of strategies of four students scoring the highest points in pre-test.

Problem-oriented teaching approach, which emphasized the importance of discussing and explaining ideas, no matter even if they are just partially constructed or immature, seemed to have a positive influence on how students communicated their thinking in written tasks. By the end of the school year there was a significant improvement on students' reasoning skills, use of heuristics and abilities to explain their thinking. Some limitations on these observations needs to be addressed: with this research design, it is not possible to assess, whether the skills would have been improved by more traditional teaching approach as well. Another challenge is linked to the test items: several of them appeared to be too easy for Finnish students and tasks were completed in a shorter time than expected. A test with a wider variety of difficulty and more items would provide more reliable information on possible development of skills and strategies.

## 6 Discussion

The study focused on exploring whether students benefitted from a problem-solving focused teaching approach, which introduced them to a general set of heuristics as a concrete tool called Problem-solving Keys. This tool worked as visual reminders of a variety of generalisable approaches for mathematical problems. The study aimed to

explore whether this kind of an active, heuristics-based teaching approach would improve fifth graders' performance and use of strategies and develop students' skills in explaining their thinking in mathematical tasks.

The analysis indicated that skills to explain thinking improved. Before the intervention, students generally relied on intuitive strategies or opted to leave the justification completely out. After getting familiar with concrete tools for general heuristics, students became more confident in expressing their ideas and justifying their strategies and were also willing to help the others by explaining solution methods. Mathematical discourse helps students not only to develop their understanding of mathematical ideas, but also to build a personal relationship with mathematics on an emotional level (D'Ambrosio & Prevost, 2008). After the intervention students' variety of heuristics increased and they were able to choose more sophisticated ones when solving different tasks. This can be interpreted as a positive development. This development was visible also in situations, in which the student still worked by implementing an erroneous strategy: in many cases, there was a shift from an intuitive approach towards a more sophisticated, yet erroneous approach. For example, an ability to base decisions on multiplication and demonstrate some understanding of the relative nature of the task can be interpreted as a step towards proportional reasoning, even though the student would not be able to expand the idea to cover the whole concept on a certain task. Findings of this study suggests that also the erroneous approaches can be viewed as hierarchical steps towards more sophisticated skills and correct reasoning. Teacher has a crucial role in recognising these small steps, for example student's transition from erroneous intuitive approaches (for example drawing) towards additive and multiplicative reasoning and emerging skills in understanding relational nature of proportional reasoning tasks.

In Polya's model (1945/1973), the last phase of "looking back" provides opportunities to assess and discuss ideas that emerged during the problem-solving process. This research underlines the importance of discussing different approaches and heuristics already during the earlier phases of problem-solving. This increases students' confidence in presenting also the partially correct ideas, which can be seen as steps forwards. Teacher's role is to make sure that students are not left with the impression that *any* answer is mathematically valid, and to guide them towards correct methods and mathematically correct language.

By teaching heuristics, students learn to solve complex word problems, reason mathematically in everyday situations and develop their thinking skills. Heuristics should be understood as general guidelines, methods, or possibilities to approach a diverse set of mathematical problems. Still, learning heuristics does not alone help students, and heuristics as such should not be reduced to learning certain techniques or sets of algorithms to choose from. Learning to describe and justify thought processes is equally important. It can be asked whether school mathematics in primary schools supports students' development in explaining their thinking, or is the focus still on finding the correct answer? This is problematic when considering the transition to secondary school, where students are expected to be able to explain their thinking by using mathematical language. Teaching approach, which guides students in justifying their ideas by using various methods, develops mathematical problem-solving skills and creates an excellent foundation for learning more complex mathematical concepts. Classroom discussions enable the teacher to make decisions on which state students benefit from teacher's guidance, and when it is more fruitful to let them find out the solution by themselves.

A few limitations of this study need to be addressed. The data for this research was collected from one sub-urban, monolingual primary school in Northern Finland. With a larger sample from Finnish schools, it would have been possible to gain more generalisable results on whether the students' performance and use of certain strategies would follow similar trends in schools in different areas. Another limitation is linked to the development of problem-solving skills and the possible effect that the teaching approach had on the results: it would have been beneficial to have the same pretest-posttest setting with a group of students without the intervention. At the point of implementing the teaching approach and collecting the data, this was not the main focus of the research, but the aim was to develop and assess the heuristics-based teaching approach, practicing teacher being also the researcher (e.g. Niemi & Nevgi, 2014). Further research is needed to understand the natural development of problem-solving skills and strategies, and whether and what kind of "out-of-the-textbook" approaches in mathematics classrooms could enhance these skills.

Students should be provided rich mathematical problems and taught a variety of problem-solving heuristics to tackle the demands of the 21st century. Mathematics should work as a tool, which would help in facing everyday situations. Even though not everyone becomes a mathematician, students' fluency as mathematical thinkers and problem solvers can be supported by paying attention in developing their skills

already in primary school. Mathematics curriculum in Finland offers flexibility to shift from arithmetic “fill-in-the-book” exercises towards a meaningful problem-solving teaching approach. Teaching mathematics through problem-solving provides opportunities to develop a wide variety of problem-solving strategies and heuristics. Problem-solving Keys are one easily accessible tool to enhance these skills.

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# Supporting argumentation in mathematics classrooms: The role of teachers' mathematical knowledge

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Reform movements in mathematics education advocate that mathematical argumentation play a central role in all classrooms. However, research shows that mathematics teachers at all grade level find it challenging to support argumentation in mathematics classrooms. This study examines the role of teachers' mathematical knowledge in teachers' support of argumentation in mathematics classroom. The study addresses a documented need for a better understanding of the relationship between mathematical knowledge for teaching and instruction by focusing on how the knowledge influences teachers' support of argumentation. The results provide insights into particular aspects of teachers' mathematical knowledge that influence teachers' support of students' development of valid mathematical arguments in mathematics classrooms and suggest implications for research and practice.

Keywords: mathematical knowledge for teaching, argumentation, mathematical arguments, collective argumentation, teacher support of argumentation

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## 1 Introduction

Reform movements in mathematics education advocate that mathematical argumentation play a central role in all classrooms. In particular, mathematics classrooms should become communities of inquiry in which students seek, formulate, and critique the validity of each other's conjectures and arguments (See e.g., National Council of Teachers of Mathematics [NCTM], 2000; CCSSM, National Governors Association Centre for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Yet, research shows that teachers find it challenging to support argumentation in mathematics classrooms (See. e.g. Ayalon & Even, 2016; Bieda, 2010). Furthermore, teachers' mathematical knowledge plays an important role in their support of this practice in mathematics classrooms (Cengiz et al., 2011; Yackel, 2002). The study examines aspects of teachers' mathematical knowledge that influence teachers' support of argumentation. There is a documented need for a better understanding of the relationship between teachers' mathematical knowledge and aspects of instruction (See e.g. Cengiz et al, 2011). This study examines the



relationship between mathematical knowledge and teachers' support of argumentation in classrooms. Processes involved in argumentation are similar to those involved in mathematical thinking. Therefore, supporting argumentation is supporting mathematical thinking, the topic of this special issue.

## 2 Theoretical Background

### 2.1 Argumentation in mathematics teaching

In the mathematics education community, argumentation is considered an important disciplinary practice that should be promoted in all classrooms. The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 2000) emphasize reasoning, proof, and communication, three essential components of argumentation. The Common Core State Standards for Mathematics (CCSSM, 2010) state that students should be able to “Construct viable arguments and critique the reasoning of others” (p. 7). There are several reasons for promoting argumentation in mathematics classrooms. Students' ability to justify claims, which is part of argumentation, is considered a key indicator of students' mathematical thinking (CCSSM, 2010). Argumentation is a natural part of doing mathematics since mathematics is a proving science and mathematical argumentation is central to proving (Ubuz, Dincer, & Bulbul, 2012). Argumentation can also help promote equitable learning opportunities in classrooms. This is because argumentation is a central construct to discourse and classroom discourse influences students' access to mathematics. Teachers can promote equity in learning by providing all students with opportunities to produce and defend their arguments in classroom discussions (Bieda, 2010).

Research on argumentation in mathematics classrooms has examined the classrooms conditions and the role of the teacher in facilitating the process (Ayalon & Even, 2016; Conner et al., 2014; Douek, 1999; Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Maher, 1998; Mueller et al., 2014; Yackel, 2002). This research shows that teachers can play a central role in supporting argumentation. They can negotiate classroom norms that foster argumentation as the core of students' mathematical activity, support students as they interact with each other to develop arguments, and supply argumentative supports (data, warrants, and backing) that are either omitted or left implicit (Yackel, 2002). When supporting students working collaboratively to develop mathematical arguments, teachers can prompt students to establish claims

and justifications, encourage them to critically consider different arguments, present to students what constitutes acceptable mathematical arguments, and model ways of constructing and confronting arguments (Ayalon & Hershkowitz, 2017).

Despite its importance for mathematical learning, the implementation of argumentation in mathematics classrooms is not common practice (Bieda, 2010; Bleiler, Thompson, & Krajcevski, 2014; Staples, Bartlo, & Thanheiser, 2012). Research shows that teachers find it challenging to incorporate this practice in classrooms (Ayalon & Even, 2016; Bieda, 2010). They find it challenging to engage students in constructing and critiquing arguments (e.g., Ayalon & Even, 2016) and their interpretations of facilitating argumentation may not be aligned with those of reformers such as assuming that mathematical argumentation can occur with relatively little scaffolding by the teacher (Kosko et al., 2014). There is a general consensus that research on teacher support of argumentation is still in its infancy and more needs to be known about teacher knowledge and practice of argumentation (Kosko et al., 2014; Mueller et al., 2014). This study addresses this issue by examining aspects of mathematical knowledge that influence teacher's support of argumentation.

## 2.2 Mathematical argumentation

Research on argumentation in educational settings frequently uses Toulmin's (1969/2003) scheme of argumentation as an analytical tool. According to this scheme, the core of an argument consists of three essential parts: *claim*, *data*, and *warrant*. The *claim* is the assertion of which an individual is trying to convince others. The *data* are the evidence that the individual presents to support the claim. The *warrant* is the explanation why the claim follows from the data. Members of a group may not be convinced that a claim follows from the data and question the validity of the warrant. In such cases, the individual may present a support or *backing* for the warrant. The model has two additional components: a modal *qualifier*, which refers to the degree of confidence about a claim, and a *rebuttal*, which refers to the conditions under which the conclusions may or may not hold. The restricted version of Toulmin's scheme is considered sufficient to analyze arguments at school level (Knipping and Reid, 2015; Krummheuer, 1995). However, Inglis et al. (2007) showed that considering the additional components can provide a more comprehensive description of individuals' argumentation and reasoning processes and helps investigate arguments similar to those made by mathematicians.

Krummheuer (1995) extended Toulmin's notion of argumentation from an individual to a collective notion by distinguishing between situations where one individual tries to convince an audience about the validity of a claim and situations where two or more individuals interact to attempt to establish a claim, which Krummheuer called collective argumentation. Collective argumentation thus becomes an interactional discursive accomplishment and an argument can no longer be analyzed solely by considering a sequence of statements that are made. The functions that various statements serve in the interaction of participating individuals become critical to making sense of the argumentation that develops. What constitutes data, warrants, and backing is no longer predetermined, but rather negotiated by the participants in the interaction. This makes collective argumentation a useful construct for analyzing mathematical activity characterized by collective problem solving (see, e.g., Whitenack and Knipping, 2002; Van Ness and Maher, 2019). In particular, this makes collective argumentation a useful construct for analyzing the teacher's role in facilitating argumentation as the teacher interacts with students to support the development of valid mathematical arguments (Yackel, 2002). In this study, teachers' support of argumentation refers to teachers' discursive role in supporting students' development of valid mathematical arguments to support their solutions as they work collaboratively on challenging mathematical problems.

### 2.3 Mathematical knowledge for argumentation

It is generally accepted in the mathematics education community that the quality of mathematical teaching depends on subject-related pedagogical knowledge that teachers bring to bear on their work and this type of knowledge goes beyond what one acquires as a student of mathematics (Adler & Davis, 2006; Ball et al, 2004; Ball, Lubienski, & Mewborn, 2001). However, there is no universal agreement on one widely-accepted framework for describing this knowledge (Petrou and Goulding, 2011). Several conceptualisations or models have been proposed over the years (See e.g., Shulman, 1986; Fennema and Franke, 1992; Rowland, 2005; Rowland, 2007; Rowland, Huckstep, & Thwaites, 2003). Petrou and Goulding (2011) provide a comprehensive review of the models focusing on their meaning, importance, limitations, implications for research and teacher development, and the political context in which they were developed. They note that the models elaborate rather than replace Shulman's (1986) well-known conceptualisation of content-related categories of teacher knowledge, particularly the categories of Subject Matter Knowledge (SMK)



and Pedagogical Content Knowledge (PCK).

One model is the Mathematical Knowledge for Teaching (MKT) framework proposed by Ball et al. (2008). The model distinguishes among three SMK subcategories. *Common Content knowledge (CCK)* is the mathematical knowledge held by people who have not taught children mathematics. *Specialized Content Knowledge (SCK)* is the mathematical knowledge specific to teaching and includes being able to examine alternative representations, provide explanations, and evaluate unconventional methods. *Knowledge at the Mathematical Horizon* is the “awareness of how mathematical topics are related over the span of mathematics included in the curriculum.” The MKT framework also distinguishes among three PCK subcategories. *Knowledge of Content and Students (KCS)* is knowledge of how students learn specific mathematical ideas and concepts, students’ common conceptions and misconceptions, and what students are likely to do in specific mathematics tasks. *Knowledge of Content and Teaching (KCT)* is knowledge of effective strategies for teaching particular content, and includes useful examples for highlighting important mathematical issues, and the advantages and disadvantages of using particular representations to teach specific ideas. There is also *Knowledge of Curriculum (KC)* which is provisionally placed in PCK category.

In this study, teachers’ mathematical knowledge refers to MKT knowledge for supporting argumentation. Research shows that having strong knowledge in MKT areas enhances teachers’ support of students’ mathematical learning (Hill et al. 2005, 2004). However, the relationship between knowledge in MKT areas and instruction remains unclear (Ball et al, 2001; Cengiz et al. 2011; Tirosh and Even 2007). This study examines the relationship between teachers’ MKT knowledge and teachers’ support of argumentation in mathematics classrooms. The focus is on identifying aspects of MKT in the areas of SCK, KCS, and KCT that help teachers support students’ development of valid mathematical arguments. The following research questions guided the study:

1. What aspects of mathematical knowledge for teaching (MKT) in the areas of SCK, KCS, and KCT support teachers in facilitating students’ development of valid mathematical arguments in collaborative problem solving?
2. How do such aspects support teachers in promoting argumentation in mathematics classrooms?



### 3 Method

#### 3.1 Research context

The three-year after-school classroom-based *Informal Mathematical Learning* project (IML) provided the context for the present study. The goal of the project was to understand how students reason in building mathematical knowledge as they worked collaboratively on challenging mathematical tasks. The project was implemented in an economically depressed urban district in the Northeast coast of the United States. Ninety-eight percent of the students were African American or Latin. Approximately twenty-four sixth-grade students, all African American or Latin, volunteered to participate in the project. During IML research sessions students worked for sixty to ninety minutes on mathematical tasks selected from several mathematical content strands including combinatorics, proportional reasoning, early algebra, and probability with dynamic software. Students worked in particular conditions: they were encouraged to work collaboratively and to always justify their solutions to problems to each other. Their contributions were encouraged and always received positively. They were asked to evaluate their claims based on whether or not they were convinced that they “made sense” and they were given extended time to work on tasks. Follow-up interviews with students were conducted after sessions to gain an in-depth understanding of the students’ reasoning.

Seven elementary school mathematics teachers participated as interns in the IML project. Their participation was part of a professional development program designed to help teachers develop knowledge to promote mathematical reasoning and justification in teaching. During the first year of the project, the teachers observed researchers lead research sessions with a class of sixth-grade students. During the second year, partner teachers led similar sessions with a new cohort of sixth-grade students, implementing the same content, while other teachers, researchers, and graduate students observed. At the end of each paired teacher implementation session, one-hour debriefing meetings were held for reflection and discussion of challenges in supporting students’ thinking.

#### 3.2 Data source

All research sessions and debriefing meetings in the IML project were videotaped and digitized. Several cameras captured students’ mathematical activity in small groups

and whole class discussions as well as teachers or researchers' exchanges and interactions with students and facilitation of conversations about students' presentations on an overhead projector for sharing of student work. One camera captured the debriefing meetings. Data for this study was selected from videos of IML student sessions led by teachers in the second year of the IML project and of debriefing meetings held at the end of the sessions and attended by teachers and researchers. Examining videos of IML sessions showing teachers' pedagogical actions and videos of debriefing meetings showing teachers reflecting on their actions is consistent Shulman's observation that the knowledge base for teaching is distinguished by "the capacity of a teacher to *transform the content knowledge he or she possesses into forms that are pedagogically powerful*" (1987, p. 15; emphasis added) and the distinction by Ball (1988) between knowing mathematics '*for yourself*' and knowing in order to *be able to help* someone else learn it (emphasis added). This suggests that mathematical knowledge for teaching is reflected both in teachers' utterances/reflections (Debriefing meetings) as well as their actions (IML sessions) while teaching. There were approximately twenty IML sessions and an equal number of follow-up debriefing meetings during each year of the project. Data for this study consisted specifically of videos of six teacher-led sessions and debriefing meetings held at the end of the sessions, all involving versions of the Tower Problem, a task that was part of the counting strand. The statement of the Four-Tall Tower Problem when choosing from two colors read as follows:

You have two colors of Unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates.

Other versions of the Tower Problem used in the IML project included the Two-tall tower problem when choosing from three colors and the three-tall tower problem when choosing from three colors. The tower problem is reasoning-rich. Students often use different strategies and types of reasoning to solve the problem (See e.g., Maher et al, 2010). This and the fact that in IML students were asked to justify their solutions to each other and to teachers/researcher helped create a learning environment for studying teacher support of argumentation. This is a case study. Stake (1994) defines an instrumental case study as a form of research where "a particular case is examined to provide insight into an issue or refinement of theory." The six teacher-led student sessions and corresponding debriefing meetings involving versions of the Tower

Problem were (the instrumental case that) was examined to gain insight into aspects of mathematical knowledge for teaching that help teachers support argumentation in mathematics classroom (issue of interest).

### 3.3 Analysis

Data analysis combined video analysis methodologies (see, e.g., Powell, Francisco and Maher, 2003; Erickson, 2006) and analytical approaches for studying argumentation (See e.g., Krummheuer, 1995; Knipping et al, 2015). The analysis had two parts corresponding to the two types of data used in this study: (1) analysis of the IML teacher-led sessions and (2) analysis of the debriefing meetings that followed the sessions. In both cases, the analysis involved several iterations of three sequential and interrelated main steps. First, all videos were viewed several times to have a sense of the data as a whole. Second, the videos were viewed again and parsed into episodes. Third, all episodes were analyzed for insights into aspects of teachers' MKT knowledge that support teachers' actions to promote argumentation. In the case of IML sessions, the episodes consisted of instances of sustained interaction between teachers and students where the teachers tried to support students in establishing claims. In the case of debriefing meetings, the episodes were instances in which teachers reflected on their interventions during IML sessions. Analysis of the episodes from IML sessions involved (1) coding students' developing arguments using Toulmin's model, (2) open coding for aspects of mathematical knowledge for teaching in the areas of SCK, KCS, and KCT reflected in teachers' actions to support argumentation and (3) describing how those aspects influenced teachers' support of argumentation, particularly in responding to or eliciting valid mathematical arguments supported by those aspects. The challenges of using the MKT framework for characterizing teachers' knowledge base have been documented in the literature. Cengiz et al (2011) found it difficult to distinguish between CCK and SCK and chose to collapse the two categories into one category: Common Content knowledge (CCK). Similarly, Petrou and Goulding (2011) noted it may be difficult to distinguish between SCK and PCK in the MKT framework. Also, the Mathematical Horizon and Knowledge of Curriculum (KC) domains remain under-conceptualized and require further refinement and investigation (Ball et al., 2008; Lesseig, 2016; Petrou and Goulding, 2011). For this reason, the study focused on the three categories of SCK, KCS, and KCT and defined them as follows:

1. SCK – knowledge of argumentation as a mathematical process, including its components, structure, and function (e.g., Toulmin’s scheme, types of arguments, valid and invalid argument and functions and roles of arguments)
2. KCS - knowledge of students’ typical conceptions and misconceptions as well as what they can do when engaging in argumentation (e.g., typical Harel and Sowder’s (2007) proof schemes that student may use to determine if an argument is convincing or not)
3. KCT – Knowledge of interventions for (1) eliciting and (2) responding to students’ arguments (e.g., how to help students transition from authoritarian or empirical justification toward more analytical types of arguments; how to challenge invalid arguments; how to support generalization of arguments)

Analysis of debriefing meetings was used to corroborate the analysis of IML sessions. The analysis of both kinds of data helped get a more accurate interpretation of teachers’ actions for supporting argumentation. All coding and interpretations were discussed within a research team until disagreements were resolved to enhance reliability.

## 4 Results

Data analysis revealed several aspects of teachers’ mathematical knowledge for teaching that support argumentation. These are described below along with how they influenced teachers’ support of students in building valid mathematical arguments.

### 4.1 Argumentation as a discursive activity

In the episode below students were working on finding towers two-tall with exactly two colors, blue (B) and yellow (Y). Martina was working with two other students. She built four towers [BY, BY, YB, YB] and continued to build more towers despite having duplicates. When the teacher asked her how many towers there were in total, Martina said, “It depends on how many blocks [sic unifix cubes] you have.” This prompted teacher to intervene:

- Marina Did you say two colors?  
 T1: Two different colors, two tall. How many towers can you build?  
 Martina: I guess it depends on how many blocks [sic unifix cubes] you have  
 T1: Well, okay. Suppose you had more blocks? Here is another one (points at a tower the student had built). You built that one. (Asks all students) What happens if she builds that one?

Students 1 and 2: They are all the same.

Student 2: (Talking to Martina) You gotta take one of each of them out, like this.  
(removes duplicates from Martina's towers and leaves only the towers BY and YB)

T1: So, you can only make two different towers, two colors, two tall. [to all students] Do you agree?

Student 1 and 2: Yes.

T1: (Asking Martina) Do you agree, Martina? (Martina nods). Right. Because what happens? Even if you had more blocks, what happens?

Martina: It is still going to be the same.

T1: Correct. So, you start building the same thing. So, it's a repeat. Good.

Using Toulmin's scheme, Martina's argument can be coded as follows: any two cubes stuck together make a tower and there can be least four towers (data). Since more towers can be built if more cubes are available (Implicit Warrant), the total number of towers that can be built depends the number of unific cubes available to choose from (claim). Martina's argument is not valid because the data in her argument includes duplicate towers, which is not consistent with the specifications of the problem since it requires that she builds *different-looking* towers. The teacher successfully challenges Martina's argument and two moves were crucial in her intervention and provide nights into the role of MKT in supporting argumentation. First, the teacher tells Martina to pretend that she has as many towers as she wants and, as Martina tries to build more towers, the teacher points at duplicates in Martina's set of towers ("Here is another one", "You built that one"). Second, the teacher tries to involve the other students in the group in examining Martina's argument by asking questions not only Martina but also to the students (e.g., "What happens if she builds that one?" and "Do you [Martina] agree [with *them*]?"). The two moves provide insights into the role of MKT in supporting argumentation. The first move shows that the teachers understood that Martina's argument is not valid because it includes incorrect data (SCK). However. this was not enough to help Martina realize the mistake in her argument. It is the second move that effectively helps Martina realize the mistake in her argument as the other students in the group tell Martina that her set of towers includes duplicates ("they are all the same") and one student even removes the duplicates from her set of towers. This highlights the importance of the view of argumentation as a social practice which emphasizes social and cultural aspects and persuasion as its main function, compared to a view of argumentation that emphasizes structural or cognitive aspects and validation as the main function of the process (SCK). In this episode, the view of argumentation as a social practice allowed the teacher to use students' collaboration in the evaluation of

a mathematical argument as a strategy for supporting argumentation (KCT). The strategy helped Martina realize abandon an invalid argument.

## 4.2 Counterexamples to challenge arguments

In IML sessions, students eventually arrived that at the correct solution that there are in total sixteen towers four-tall when choosing from two colors. The teacher challenged the students to justify the solution. One student, Gabriel (Gabe) built four groups of four towers and then said that there are sixteen towers in total because “four times four is sixteen,” (See [Figure 1](#)). When the teacher asked the student why he said “four times four?” the student said “you can divide the sixteen towers into groups of four towers each.” The teacher was not convinced by the student’s explanation, but did not know how to challenge it and walked away. In the debriefing meeting that followed the session, the teacher shared with the audience her difficulty in challenging the 4x4 argument admitting that she did not know how to “elicit the convincing argument” from the student:

T1: Gabe said, “Sixteen divided by four is four.” I am like, “Well, what does that have to do with what we are doing?” *So, the question I have as a facilitator is, what do I do in order to elicit the convincing argument?* Because even with Yonnie, he is getting at a point where he is getting annoyed with me because I keep saying, “How do you know?”

One teacher in the audience suggested asking the students to write their solutions on posters and then share them and discuss in class. However, the researcher who was facilitating the meeting proposed a different idea. She suggested first asking the students to predict how many towers there would be three-tall when choosing from two colors and then asking them to investigate empirically if their prediction was correct. The researcher explained how she thought the students’ reasoning would unfold. The students would predict nine towers by analogy with the 4x4 argument and then, when trying to build them, they would not find nine towers and would find only eight. They would also notice that every tower has an opposite-looking tower and would conclude that the total number of towers had to be even and would abandon their prediction:

Researcher: The way I frequently address that is to say, “*Okay, how many [towers] do you think there would be if they were just three tall with two colors to choose from?*” And Yonny [student] is going to say “Nine.” And I say, “Hm ... that does work with your prediction. How are you going to test that one out?” And Yonny is going to say, “I guess I can build them.” Then I say, “But don’t



mess up your fours [four-tall towers they built].” Make sure they don’t destroy their four ones in order to do the three ones. And then when they can’t find them [the nine predicted towers] ...there is a little bit of disequilibrium...Also, “what do you think it’s going to be for five?” Then they’re going to say “Twenty-five.” *Many people say, you know, “We know it’s going to be even. So, it can’t be nine. So, it must be one less.” People of all ages stay with the four by four but modify it because we know there are opposites. It’s got to be an even number of them. Eight is one fewer than nine. So, it’s got to be twenty-four. I would keep pushing them in all the ways you are thinking about. That is just my suggestion.*

In the following session, the teachers implemented the researcher’s suggestion and events unfolded exactly as the researcher had predicted. Several students predicted that there would be “nine” towers three-tall when choosing from two colors. One student, Mohamed, said, “Maybe nine because three times three equal nine.” However, the students could not find 9 towers. They found only eight towers. Also, Martina, the student in the previous episode, noticed that every tower had a “double” and concluded that that there could not be nine towers because the total number of towers had to be an even number:

Martina: I said if it was 9 there would be like double of them because of the opposite of one another. Like this one blue (BBB) and this one is yellow (YYY) there would be another one. Except there would be an opposite. So, it has to be an even number.

Based on Toulmin’s scheme, Gabe’s argument can be coded as follow: For towers four-tall there are four groups of four towers each (data). Therefore, the total number of towers must be sixteen towers (claim) because the total number of towers must “height x height” (implicit warrant). The implicit warrant in Gabe’s argument is supported by the students’ prediction that there would be nine towers three-tall when choosing from two colors because “three times three equal nine,” where “three” is the height of the towers. However, is not valid as a general warrant as it did not work for the three-tall towers problem when choosing from two-colors. This episode highlights the importance of counterexamples in supporting argumentation. The three-tall towers problem when choosing from two colors served as a counter-example to the “height times height” general warrant implicit in the “4x4” and “3x3” arguments. If the warrant was correct, there would be  $3 \times 3 = 9$  towers three-tall when choosing from two colors. However, the students could not find nine towers and could not support the “height x height” warrant implicit in the 4x4 argument. Knowledge of counterexamples that challenge particular arguments can be considered an example

of SCK. This episode shows that such knowledge can help support argumentation by challenging invalid mathematical warrants (KCT).



Figure 1. Gabe (on left) built for groups of four towers each when choosing from two colors

### 4.3 Knowledge of Students' argumentative strategies

In the 4x4 argument above the researcher introduced her suggestion by saying “*The way I frequently address that is to say...*”, “*Yonny is going to say...*,” and “*Many people say...*” This suggests that the researcher was using her knowledge of how students reason when working on the tower problem to come up with the suggestion. The researcher knew that students often came up with invalid warrants such as the “height x height” embedded in the 4x4 argument and designed interventions to challenge them using counterexamples such as the three-tall tower problem when choosing from two colors. In contrast, in another episode in which students were asked to build towers three-tall when choosing from three colors, Yonny, the student mentioned in the 4x4 argument above, used a reasoning-by-cases strategy and a “diagonal strategy” to prove that he has built all towers within the cases (See diagonals in [Figure 3](#)). An example of the application of the diagonal strategy to prove that all towers with three reds and one yellow (3R and 1Y) have been found is to show that the yellow cube has occupied all possible positions in the tower forming a (yellow) diagonal. Teacher T5 explained during the debriefing meeting that he was familiar with the use of the strategy when building towers from two colors, but was surprised to see it being used with towers with three colors:

T5: I don't know why in my mind I didn't think it would work when I went around to see his. At first, I didn't say anything to him. I've learned that. But I just looked at it and asked him to explain it, but now it makes sense.

The statement suggests that not knowing that the diagonal strategy could be used with towers with three colors constrained the teacher's support of Yonny's reasoning process (*I didn't say anything to him.... But I just looked at it and asked him to explain it*). This and the way the researcher introduced her suggestion in 4x4 episode highlight the importance of KCS in supporting argumentation. It shows that having knowledge of argumentative strategies that students are likely to use when making argumentation (KCS) can help teachers design effective strategies for support argumentation. In the 4x4 argument, knowing that students could use the "height x height" argument helped the researcher come up with a counterexample to challenge the argument. However, T5 did not know that students could use the diagonal strategy with three-color towers and this limited the teacher's ability to support a student in developing of a valid argument based on this strategy. Overall, knowledge of students' typical reasoning or argumentation support strategies for promoting argumentation that build on students' argumentative reasoning (KCT). The three-tall tower problem when choosing from two colors was carefully designed task to be a counterexample to the "height x height" argument in 4x4 episode.

#### 4.4 Representation

While working on finding all towers 3-three cubes tall when choosing from three colors, Yonny came up with the tower arrangement displayed in [Figure 2](#). The arrangement shows "opposite" groups of towers (i.e., towers in one group are opposites of towers in the other group) and diagonal lines in different colors running through all groups except the groups of single-color and three-color towers. Yonny told the teacher that there were 27 towers in total because "I can't find [any more of] them." The teacher said, "that's not a proof" and Yonny responded, "I used opposites" and explained his idea using the diagonal lines:

- T1: Wait. What do you mean? So, these are opposites?  
 Yonny: Yeah.  
 T1: *Explain it to me why?*  
 Yonny: Because like I said before. You got the *yellow in a little line* here [traces the yellow diagonal in towers 2R1Y]. You got the *red in the little line* here [traces a red diagonal in the opposite group of towers]  
 T1: *What do you call that line?*  
 Yonny: *A diagonal line*  
 T1: Ok, so you are saying there is a yellow in this diagonal [towers 2R1Y] and a red in this diagonal [towers 2Y1R]? So, what does that mean?  
 Yonny: *They are opposites*. So, you got yellow and the red on both sides [of the diagonal]

The teacher turned her attention to the pairs of towers with three colors in the arrangement. The teacher started to put the towers together and then stopped and asked if the towers could form a group. Yonny said “No. because there is no other similar [*sic* opposite group],” indicating that the group would not have an opposite group. The teacher then turned her attention to the diagonals and pointed out that there were no diagonals in the group of towers with three colors. Yonny stared at the towers for a little bit and then all of a sudden had an “aha” moment. He reorganized the towers into two opposite groups of three towers each and revealed red diagonals running in opposite directions in the groups (Figure 3):

- T1: Would you put these [towers with three colors] together? Are they similar in any?
- Yonny: No because there is no other similar [i.e. Opposite group] [puts the towers back into pairs of opposite towers]
- T1: Well, ok. All right. I see why these [YYY; BBB; RRR] would go together. Tanisha [student seating at the same table], you see his diagonals here? He’s paired them up in these groups where he has these diagonals going down? [Turns back to Yonny] What about these here [towers with three colors]?
- Yonny: Those no... oooh. I think I got something. I think I got something. Oh. I am so smart. Like that [reorganizes the towers with 3 colors in two groups of three towers: BYR; BRY; RYB and RBY; YRB; YYB. A red diagonal running through each group]. Like that? Cause see... Oh shoot. See [explains how the groups are opposites] you got the two yellows and the [two] blues switch and you got these [red cubes] going in the same diagonal. You got these and blue right here and you got.
- T1: [Staring at the towers with three colors] Well, I see the diagonal here [in one of the groups] but I don’t see [a diagonal there in the other group...]. Ooohhh it [the diagonal run] in the opposite way. Cool. Hum. Hum. Very cool.

Yonny presents a proof-by-cases argument. He built 9 groups/cases of towers with 3 towers each (data). Since he believes he has *all possible groups and all towers in each group* (warrant), he concludes, by the proof-by-cases argument (implicit backing), that there must be 27 towers three-tall in total (claim). The “opposites” strategy helps Yonny account for all cases/groups and the diagonals assures him that he has all towers in each case/group. However, Yonny does not initially apply the “opposite groups” and the “diagonal” strategies consistently across the arrangement. The towers with three colors are not (1) organized in opposite groups and (2) there are no diagonals running through them as is the case with other towers in the arrangement. The teacher’s intervention helps Yonny addresses these challenges and

it shows the importance of attending to how mathematical arguments may be represented in supporting argumentation (SCK). During the episode, the teacher pays close attention to the tower representation. This allows the teacher to see that towers with three colors are organized in opposite groups and there are no diagonals running through them as the other towers in the arrangement. The teacher then challenges Yonny organize the towers as a group (*Would you put these together? Are they similar in any?*) and to show diagonals running through the towers (*What about these here [where are the diagonals]?*). Yonny addresses these challenges successfully and is finally able to apply his proof-by-cases argument consistently to the entire tower arrangement. The previous episode showed that understanding students' arguments is key for supporting argumentation. This episode shows that attention to students' representations can help teachers identify and find ways to best support students' arguments (KCT).



Figure 2. Yonny 's initial arrangement of towers for his solution to the 3-tal 3-colors problem



Figure 3. Yonny 's final arrangement of towers for his solution to the 3-tal 3-colors problem

## 4.5 Challenging arguments based on abductive reasoning

In IML sessions, there were several instances in which students justified their solutions by simply describing the models they were able to build to solve the tower problem. In the 4x4 argument above Gabe argues that there are in total sixteen towers four-tall when choosing from two colours because he built a model with four groups of four towers. In example below also involving the four-tall tower problem when choosing from two colors, James and Tanisha built four pairs of opposite towers and argued that there are eight towers in total because “4x2” equals eight:

- T1: So, how can you prove to me that you have all of them?  
 James: I was thinking that *you have to multiply four by two because there are four cubes in a tower and there are two colors*. I mean, you have to multiply the height by the [number of] colors [in a tower]  
 Tanisha: And I said that’s how you can find out how many towers we got. You can say *two [opposite towers] times four [times] equals eight [towers]*.

James’ and Tanisha’s explanations simply describe the models they built. They built four pairs of opposite towers, which equals 4x2 or eight towers. When the teacher helped them see that they could build at least two more towers (YYYY and RRRR), bringing the total number of towers to ten, James said that the two extra towers “don’t count’ and Tanisha said “You will do five times two:”

- T1: Now you agree that there are ten (towers). But *what happens to that two times four is eight and four times two is eight*, that mathematical thing that you were talking about?  
 Tanisha: [Reorganizes her towers into five pairs of opposite towers] I get the same. Because *you still can do it my way*, but it will just be five on the side and two. *You will do five times two*.  
 James: Now I am saying that these two [the extra towers], they are the same colors. They really don’t count.

James and Tanisha continue to present explanations that describe the models that they built. James says that the two extra towers “don’t count,” which preserves his original explanation by applying it to the group of towers with two colors. Tanisha says “you still can do it my way...*You will do five times two*,” which is simply a way of counting the new set of five pairs of opposite towers that she was able to build with the addition of two extra towers. The students’ emphasis on models that they were able to build suggests that the warrant supporting their arguments is empirical. For example, the 4x2 argument can be coded as follows using Toulmin’s’ scheme: There



are be at least four pairs of opposite towers (data). Since no more towers *were found* despite trying (implicit empirical warrant), there must be only eight towers in total (claim). The empirical warrant is evident in Tanisha's response "you will do five times two" when she finds out that there can be two extra towers. She adjusts her response to the *new* set of towers that she has been able to find.

The 4xx and 4x2 episode shows the importance of knowing how to challenge abductive forms of reasoning (KCT). In abductive reasoning students present the *best or most plausible* explanation to support their claims and it can be the model they were able to build if it supports the claim that they want to make. In the 4x2 episode, four pairs of opposite towers (or 4x2) does equal to eight towers which the students believe to be the total number of towers because they could not find more towers. The challenge in countering arguments based on abductive reasoning is that it can be difficult to cause a cognitive disequilibrium in the students reasoning because the explanations presented are plausible or fit the argument that they are trying to make. However, the 4x4 episode above may suggest ways for challenging this type of reasoning. The teacher uses a suggestion from a researcher to ask the students to empirically investigate the validity of their prediction that there would be  $3 \times 3 = 9$  towers three-tall when choosing from two colors based in their 4x4 model. The prediction does not hold which challenges the warrant in the 4x4 argument. This suggests that asking students to (1) empirically investigate the validity of general warrant that follow from their argument and/or (2) using counterexamples (the two-tall three-color tower problem) can help successfully challenge arguments based abductive forms of reasoning.

## 5 Discussion

This study examined the relationship between subject-related pedagogical knowledge and mathematical instruction using the Mathematical Knowledge for Teaching (MKT) framework. Specifically, the study examined how aspects of mathematical knowledge for teaching in the areas of SCK, KCS, and KCT that support teachers in promoting argumentation in mathematics classrooms. The results reveal several aspects including (1) knowledge of counterexamples, (2) a view of argumentation as a discursive process, (3) knowledge of (students') typical argumentative strategies, (4) representation of mathematical arguments and (5) knowledge how to challenge arguments based on abductive forms of reasoning. These aspects can help teachers elicit valid mathematical arguments from students in collective problem solving. The

results offer important insights into teacher knowledge and practice of argumentation in mathematics classrooms.

There are several definitions of argumentation, which reflects different perspectives on argumentation and its function (Schwarz et Hershkowitz, 2010). Some perspectives emphasize cognitive and structural aspects and validation as the main function of argumentation. Other perspectives empathize social and discursive aspects and persuasion as the main function of the argumentation (See., van Eemeren et al, 1996; Krummheuer, 1995; Baker, 2003). In a study that examined how teachers select tasks to promote argumentation, Ayalon and Hershkowitz (2017) found that teachers emphasized socio-cultural aspects of argumentation including student-teacher interactions and collective processes of argumentation (where arguments are constructed and critiqued). Ayalon and Hershkowitz used this finding to recommend incorporating this dimension into current frameworks for examining the effectiveness of textbook tasks for promoting argumentation, which they argue tend to focus mainly on structural and cognitive aspects of argumentation. The results of this study provide further support for an emphasis on the socio-cultural view of argumentation, showing that it can help teachers support argumentation in mathematics classrooms by allowing them to engage students in mathematical discussions that help challenge invalid arguments.

Studies show that introducing representations or contexts that are familiar to students and using counter-examples are two of the least frequent instructional actions in mathematics classrooms (Cengiz et al, 2011). In this study, a teacher was able to identify elements of an emerging reasoning-by-cases argument by examining a student's tower representation (groups/cases of towers and a strategy for proving that all towers in a case were found) and then use it to challenge the student to apply the argument consistently across all cases and complete the argument. In another episode, the same teacher used a counterexample to successfully challenge an invalid warrant in a student's argument. This shows that students' representations and counterexamples can be important tools for supporting argumentation in mathematics classrooms and need to be emphasized more in mathematical instruction.

In many episodes in this study, supporting argumentation involved attending to and building on students' particular reasoning or arguments. A researcher suggested a counterexample that was used to challenge an invalid warrant based on her knowledge of how students reason when working on the Tower Problem. The attempt

by teacher T5 to support a student, Yonny, in building an argument to support a solution to the three-color tower problem was constrained by the teacher's lack of familiarity with the use of the "diagonal" strategy in the problem to prove that all towers of a particular were found. In contrast, teacher T1 successfully helped the student develop a complete proof-by-cases argument after identifying aspects of the argument in the students' tower representation. These episodes highlight the importance of teachers' understanding of students' mathematical reasoning in supporting argumentation and suggest that the supporting argumentation is more likely to be effective when it builds on students' argumentative reasoning.

The results of this study show the challenges of countering arguments based on abductive forms of reasoning. In mathematics classrooms this type of reasoning is common and one way students often engage in such arguments is by offering explanations that simply describe the models that they built to solve a problem. The challenge in countering such arguments is the difficulty to cause cognitive disequilibrium in students' thinking because the models often support the solution. The results of this study suggest that teachers can challenge abductive types of arguments by inviting students to empirically investigate the validity of general warrant that support the particular argument through counterexamples. As students find out that the general warrant is not valid, they begin to question the validity of their argument.

The results may emphasize individual aspects of MKT knowledge in the areas of SCK, KCS, and KCT that help teachers support argumentation in mathematics classrooms. However, as some episodes suggest, a combination of aspects in the three areas is more likely to help teachers successfully support argumentation in mathematics classrooms. Being able to make sense of students' mathematical reasoning and arguments (KCS) can help teachers design appropriate interventions for supporting the students' development of valid arguments (KCT). However, making sense of students' mathematical reasoning and arguments may require teachers' understanding of and skills in argumentation as a mathematical process (SCK). The episode involving the counterexample helps illustrate the idea. The researcher suggested the counterexample (KCT) based on her knowledge from experience of students' reasoning when working on the tower problem (KCS) and also her understanding of the general warrant that was implicit in the student's 4x4 argument (SCK). The combination of aspects from the three knowledge categories helped successfully challenge the argument.

The results of this study have implications for practice and research. First, the results emphasize the importance of building on students' mathematical reasoning in supporting argumentation. This suggests that professional development programs need to pay particular attention to teachers' understanding of students' mathematical reasoning and argumentation if they are to prepare teachers to support the practice more effectively in mathematics classrooms. Second, teachers can build knowledge of students' mathematical reasoning and argumentation from experience. However, in this study a researcher came up with the counterexample used to challenge a student's argument based on the researcher's knowledge of how students' reason when engaging in the tower problem. This suggests that close collaboration between practitioners and researchers can help create important synergies for generating important knowledge of students' mathematical reasoning that can help teachers support argumentation in mathematics classrooms. Third, the results emphasize the importance of teachers paying more attention to students' mathematical representations and using more counter-examples in instruction. These are two of the least frequent instructional actions in mathematics classrooms. Yet, this study shows that they can help for support argumentation in classrooms. Finally, this study suggests that a potentially important area for research could be the extent to which professional development models such as the IML project involving a close collaboration between teachers and researchers in after-school settings can be successful in supporting teachers in building the knowledge they need to support argumentation and thoughtful mathematical activity in mathematics classrooms.

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# Languaging and conceptual understanding in engineering mathematics

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The ability to apply mathematical concepts and procedures in relevant contexts in engineering subjects sets the fundamental basis for the mathematics competencies in engineering education. Among the plethora of digital techniques and tools arises a question: Do the students gain a deep and conceptual enough understanding of mathematics that they are able to apply mathematical concepts in engineering studies? This paper introduces the use of languaging exercises in the engineering mathematics course 'Differential Calculus' during the spring semester 2020, at Tampere University of Applied Sciences, TAMK. In this study, the students' conceptual understanding and learning of differential calculus is researched. In the learning process, the languaging method is used to deepen the conceptual understanding of the concepts of differential calculus. Pre-test/post-test setup was used to see the possible gain in conceptual understanding. During the course, students did online assignments, which included languaging exercises. Students described the concepts of differential calculus using natural language, pictures, or a combination of them. The students were also asked to fill in a self-evaluation form to collect their perception of their own knowledge of mathematical skills. Mid-term and final exams summarized the acquired knowledge. The study aimed to enhance the learning outcomes and to gain a deeper understanding of mathematical concepts by exploiting the languaging method.

Keywords: languaging, mathematics, engineering, conceptual understanding

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## 1 Introduction

The way we teach and learn mathematics has changed in the past few decades. Technological tools have enriched the resources available for teaching and learning through 'computer aided' devices, through appropriate software, and through learning platforms. Today's students are more accustomed to learning with the help or the aid of state-of-the-art technologies. Using tools and calculators to solve exercises speeds up the calculations and provides usually more accurate results. Among this plethora of digital techniques and tools arises several questions: Do the students gain a deep and conceptual enough understanding of mathematics that they are able to apply mathematical concepts in engineering studies? Do the students just master the tools without understanding what they are doing and what does the result mean, e.g. I solved a derivative – but what does it actually mean?



According to the literature (Woods et al., 1997; Bok, 2006) and the authors' own experience, students seem to be able to mechanically repeat the known procedures to solve problems, to carry out assignments quite well – but they do not necessarily learn to think. In engineering mathematics, the foundation of learning mainly evolves from thorough understanding of mathematical concepts and the ability of exploiting abstractions to solve engineering problems. The fundamental aim of mathematics in engineering education is mathematics competencies, which means the ability to apply mathematical concepts and procedures in relevant contexts (Alpers et al., 2013).

This paper presents how the method of languaging is implemented to clarify mathematical concepts and to promote deeper learning. By making concepts of the subject more concrete to students, the aim is to clarify mathematical expressions and lead to the students' better understanding of the subject.

In a previous study it was shown that languaging exercises do have an effect on knowledge of the theory (Rinneheimo et al., 2020). In that study, an independent-samples t-test was conducted to compare if the students gain a better knowledge of the theory with the help of the languaging exercises. As a result, there was a significant difference in the scores for using the languaging exercises during the course and not using the languaging exercises during the course.

In this paper, a deeper view on the understanding of the concepts with the help of languaging method and self-evaluation has been taken. This paper focuses on promoting higher understanding of concepts by utilizing languaging exercises.

## 2 Theoretical background

### 2.1 Mathematical thinking and languaging exercises

Mathematical thinking is usually expressed with symbols, expressions, calculations etc. (by symbolic language). Languaging in mathematics refers to expressing a student's mathematical thinking through different ways, such as writing/orally using natural language, by pictures, or by a combination of these (by natural language, mathematical symbolic language, or pictorial language) (Joutsenlahti, 2010; Joutsenlahti et al., 2013). O'Halloran (2015) has presented that language assists in reasoning the mathematical process and its results. Symbols describe mathematical relations and visuals present images to concretize mathematical relations (O'Halloran, 2015). In this study, the languaging of mathematics forms an approach to making meanings of mathematical concepts and procedures. This meaning-making

process enables the students' mathematical thinking and knowledge construction (Morgan, 2001; Schleppegrell, 2010; Joutsenlahti et al., 2017). Solving a mathematical exercise or presenting the solution to a mathematical exercise by using different languages assists a student to organize their own mathematical thinking and eventually gaining a better understanding of that mathematical concept or procedure (Joutsenlahti et al., 2015; Joutsenlahti et al., 2017).

There are different types of languaging exercises and the exercises used in this study are presented in Table 1.

**Table 1.** Languaging exercises (Joutsenlahti, 2010; Joutsenlahti et al., 2013; Joutsenlahti et al., 2014)


Type of the languaging exercises	Description of the exercises
Argumentation of the solution.	Student writes or selects a natural language explanation for the solution in place of using symbolic language (or vice versa). Pictorial language could also be used.
Explaining in your own words.	Student provides an explanation by using natural language.
Adding missing parts of the solutions	The problem solution is uncompleted, and the student adds the missing parts.
Seeking errors.	Student has to find errors or missing items in the given solution and to correct the errors.

Some examples of the languaging exercises used in the Differential Calculus course are presented in Figures 1 – 3. In Figure 1 is presented two languaging exercises where student interpreted the graph. From the graph of the function  $h(t)$  (height  $h$  (m) is a function of time  $t$  (s)) the students were asked to explain in their own words: 1.) what is the difference between the markings  $h(1)$  and  $h'(1)$ , 2.) how would they define the derivate for the function at the point  $t = 3$  graphically, numerically and symbolically. They also needed to think about the unit for each reply. From the graph of the function  $f(x)$  the students were asked a) what is the average rate of change of the function  $f(x)$  from  $x = 0$  to  $x = 3$  and b) what is the rate of change of the function  $f(x)$  at the instant that  $x = -1$  and  $x = 1$ .

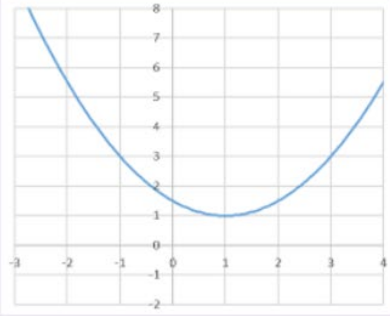
Kun pieni kappale ammutaan maanpinnalta pystysuoraan ylöspäin alkunopeudella 18 m/s, sen korkeuden  $h$  riippuvuutta ajasta  $t$  kuvaa likimain funktio

$$h(t) = 18 \frac{\text{m}}{\text{s}} \cdot t - 4,9 \frac{\text{m}}{\text{s}^2} \cdot t^2.$$

Funktioyhtälössä muuttujana on aika  $t$  ja funktion arvo on korkeus  $h$ . Ohessa on myös funktion kuvaaja.



Ohessa on funktion  $y = f(x)$  kuvaaja. Määritä kuvaajan avulla



(käytä desimaaliluvuissa desimaalipistettä)

a) funktion  $f$  keskimääräinen muutosnopeus välillä  $x \in [0, 3]$

b) funktion  $f$  muutosnopeus eli derivaatta kohdissa  $x = -1$  ja  $x = 1$ .

$f'(-1) =$

$f'(1) =$

Figure 1. Examples of the languaging exercise interpreting the graph.

In Figure 2 is an example of the languaging exercise “Seeking errors”. Students have to find errors or missing items in the given solution and to correct the errors. The exercise has been modified from task 10 of the longer mathematics course matriculation exam from spring 2017.

Tiedetään, että  $h(x) = g(f(x))$ ,  $f(x) = e^x$  ja  $g(x) = 2x^2 + 1$ . Kaksi opiskelijaa laskevat derivaatan  $h'(x)$  seuraavalla tavalla:

Opiskelijan 1 ratkaisu:	Opiskelijan 2 ratkaisu:
$f(x) = e^x$ $g'(x) = 4x$ joten $h'(x) = g'(f(x)) = 4e^x$	$h(x) = g(f(x)) = 2(e^x)^2 + 1 = 2e^{x^2} + 1$ $h'(x) = 2e^{x^2} \cdot (2x)$ joten $h'(x) = 4xe^{x^2}$

Kolmas opiskelija saa laskimella vastaukseksi  $4e^{2x}$ .

a) Kenen vastaus on oikein?

b) Etsi väorien ratkaisujen virheet ja selitä omin sanoin korjatut ratkaisut.

Figure 2. An example of the languaging exercise seeking errors (Ylioppilastutkintolautakunta, 2017).

In [Figure 3](#) is a part of the languaging exercise “Adding missing parts of the solutions”. In this kind of exercise, the problem solution is uncompleted and the students add the missing parts.

Erään kondensaattorin varaus ajan funktiona on  $q(t) = 5 \cdot 10^{-6} \text{C} \cdot \sin\left(\frac{100\pi}{\text{s}} t\right)$ .

Laske virranvoimakkuus  $i(t) = \frac{dq}{dt}$  ja virran muutosnopeus  $\frac{di}{dt}$  ajanhetkellä  $t = 0,9 \text{ s}$ .

*Ratkaisu:*

Lasketaan virranvoimakkuus ajanhetkellä  $t = 0,9 \text{ s}$ :

Derivoi funktio  $q(t) = 5 \cdot 10^{-6} \text{C} \cdot \sin\left(\frac{100\pi}{\text{s}} t\right)$

$i(t) = \frac{dq}{dt} = q'(t) =$

Figure 3. An example of languaging exercise adding missing parts.

The use of languaging has given good results in mathematics education (Joutsenlahti et al., [2013](#); Joutsenlahti et al., [2014](#); Sarikka, [2014](#); Joutsenlahti et al., [2016](#)). Languaging exercises make the student think about what they are doing, not only mechanically calculate the exercise (Rinneheimo et al., [2019](#)). One challenge of mathematics teaching is how to describe mathematical thinking and how to make it visible. The languaging exercises enable making the students’ mathematical thinking processes visible and also support the development of these processes (Joutsenlahti et al., [2017](#)).

Hiebert and Lefevre ([1986](#)) divided the mathematical knowledge to conceptual knowledge and procedural knowledge. Conceptual knowledge has been defined as understanding of the principles and relationships that underlie a domain, and procedural knowledge consists of the symbol representation system of mathematics and the algorithms and rules for completing mathematical tasks (Hiebert et al., [1986](#)). Kilpatrick, Swafford and Findell ([2001](#)) described students’ mathematical proficiency with five components as follows:

- conceptual understanding – comprehension of mathematical concepts
- procedural fluency – the ability for flexible, efficient, accurate and appropriate calculation
- strategic competence – problem solving



- adaptive reasoning – ability for logical thinking, reflection, explanation and justification
- productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

These five components of the mathematical proficiency can be seen as one way of describing the features of the mathematics. This study focuses on the skills’ conceptual understanding and procedural fluency as follows: the student has the ability to use mathematical concepts in the right context and manages the procedures behind the concepts. In this study, these skills are discussed as conceptual understanding and the capability has been studied with the languaging exercises, as illustrated in [Figure 4](#).

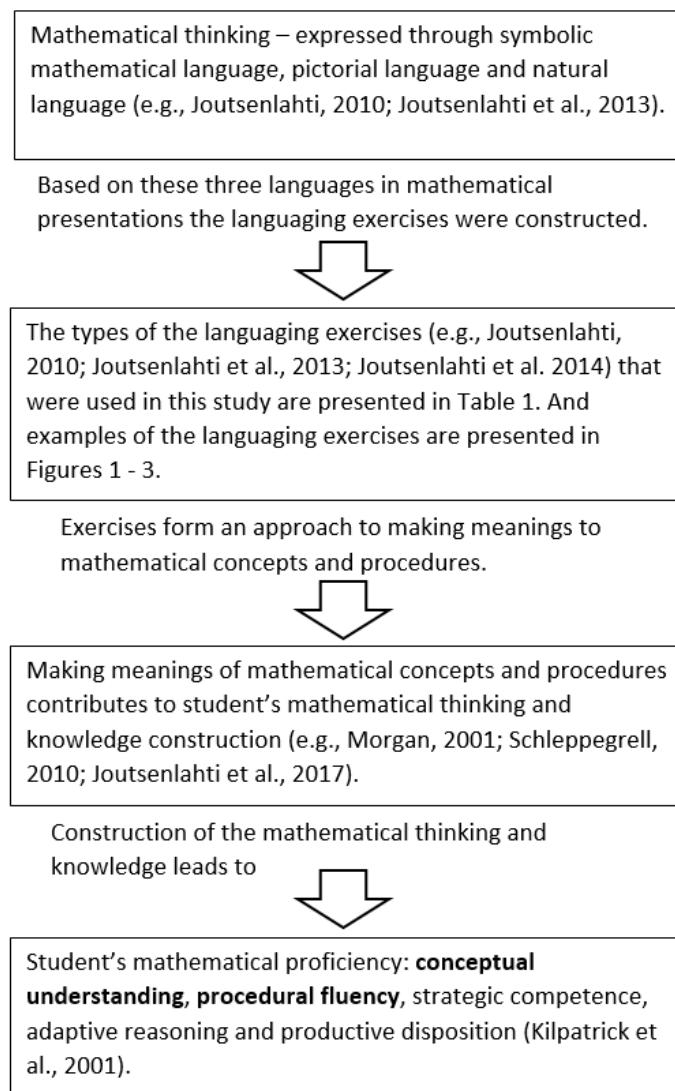


Figure 4. Building the conceptual understanding.

The languaging exercises have been formed using three languages (Joutsenlahti, 2010; Joutsenlahti et al., 2013). The exercises form an approach to making meanings of mathematical concepts and procedures (Morgan, 2001; Schleppegrell, 2010; Joutsenlahti et al., 2017), which contributes to conceptual understanding (Kilpatrick et al., 2001).

## 2.2 Meaning making and conceptual understanding

The purpose of using languaging exercises that express a student's mathematical thinking through three languages (natural language, mathematical symbolic language, and/or pictorial language), is to develop the student's own meaning making process and lead to the conceptual understanding. Boudon (2016) pointed out in his study that writing mathematics does not only strengthen the student's conceptual understanding, but can also develop their ability to communicate the meaning of such concepts. According to Morgan (2001), writing and the use of natural language in the solutions of mathematical exercises develop conceptual understanding, the attitudes of the learners towards mathematics improved, and they also facilitate the assessment work of teacher.

Also, according to Moschkovich (2015), explaining meanings, constructing arguments and justifying procedures leads to conceptual understanding. Research has shown that the use of natural language and drawings helps most students in solving mathematical exercises (Joutsenlahti et al., 2016). Languaging exercises and presenting mathematics in writing enables a student to structure and clarify their mathematical thinking (Joutsenlahti, 2010; Kangas et al., 2011).

## 3 Research process

### 3.1 Research questions

In this study, the students' conceptual understanding of differential calculus concepts is researched, and the capability has been studied with the languaging exercises. In this article, we concentrate on the following research questions:

1. How does the students' languaging ability develop throughout a course?
2. How did the mathematical languaging clarify mathematical expressions?
3. How did develop the conceptual understanding?

In the following chapter, we present the data collection process and the analysis of the data. The key idea in the teaching process and data collection was collect data from several sources during the whole course.

### 3.2 Data collection and analysis

This paper introduces the use of languaging exercises in the engineering mathematics course ‘Differential Calculus’ taught at Bachelor’s level during the spring semester 2020 at TAMK. In this study, there were two engineering student groups and the number of active students was altogether 64. Course materials were a book, a formula book, a symbolic calculator and as additional material, online exercises and timetable in Moodle learning platform.

The data was gathered from the several sources:

At first pre-test/post-test setup was used to see the possible gain in conceptual understanding. In the tests, students described, by natural language or by interpreting a graph, the concepts of differential calculus.

Secondly during the course, the students had six compulsory online assignments to be completed as homework. These assignments were prepared by using different question types in Moodle and most of the exercises in these online assignments were languaging exercises. This study compiles 14 languaging exercises from these online assignments. The topics of the assignments used in this study were graphical, numerical, and symbolic differentiation, and applied exercises. In the exercises, the students were asked to explain course concepts in their own words, or to seek errors and explain in their own words the correction to the error. Also, students interpreted graphs and in some exercises the solution to the problem was explained with natural language and the student was asked to complete or select from the list the missing calculations or symbolic presentations. Examples of the used languaging exercises are presented in Figures 1–3.

Students were also asked to fill in a detailed self-evaluation form weekly to collect their perception of mastery of that week’s topics. In the form, each week’s learning objects were described using natural language. The then students typed a letter a - d to the cell according to their perception of the mastery of the topics (a = green: I have learnt this so well that I could teach it to my peers. b = blue: I feel I understand this topic. c = orange: I think I have understood this partially, but it is partially unclear. d = red: I need more practice to understand this.). Part of the form is presented in [Figure 5](#).

Learning objectives of the week	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6	Student 7	Student 8	Student 9	Student 10	Student 11	Student 12
<b>Week 1</b>												
Item 1	Green	Blue	Blue	Blue	Green	Orange	Green	Blue	Blue	Blue	Blue	Orange
Item 1	Green	Blue	Blue	Blue	Blue	Orange	Green	Blue	Blue	Blue	Blue	Orange
Item 2	Green	Blue	Blue	Green	Green	Orange	Green	Blue	Green	Green	Blue	Orange
Item 3	Green	Green	Green	Blue	Blue	Orange	Green	Orange	Blue	Blue	Blue	Red
Item 4	Blue	Green	Green	Orange	Blue	Orange	Green	Orange	Blue	Orange	Orange	Red
Item 5	Green	Green	Green	Orange	Blue	Orange	Green	Blue	Blue	Orange	Red	Orange

Figure 5. Self-evaluation form (Peura, 2018).

During the course there were two exams (mid-term and final), which summarized the acquired knowledge. The first exam contained mechanical calculations, such as differentiate the given function, and a languaging exercise, which asked the students to interpret a graph. The second exam also contained a languaging exercise, where students explained, with their own words, mathematical concepts of the course. The second exam mainly consisted of applied exercises where the students first needed to invent the mathematical model of the assignment and then to solve it. The data collection is summarized in Table 2.

Table 2. Collection of the data.

Data sources	Languaging exercises	N
1) Pre-test/post-test	6 (Figure 6, in chapter 4.1)	53
2) Online assignments	14	64
3) Self-evaluation form	In the form each week's learning objects were described using natural language (in Figure 5 is part of the form).	59
4) Exams	mid-term: included languaging exercise, which asked the student to interpret a graph final: included languaging exercise, where students explained with their own words' mathematical concepts of the course	64

The data were analyzed by mixed methods. The MS Excel program was used for typical statistical analysis (e.g., in comparing distributions, arithmetic mean, variation, median, correlation, frequencies). The qualitative analysis was made by theory guided content analysis (e.g., categorizations). Classification into the four categories was used while analyzing the students' answers to pre-test/post-test, online assignments, and exam replies as follows: wrong/do not know (0 points), just a little

right/only some idea of the task (1 point), partly correct (2 points) and correct (3 points). Self-evaluation form's replies were categorized as follows: 0 = I need more practice to understand this, 1 = I think I have understood this partially, but it is partially unclear, 2 = I feel I understand this topic and 3 = I have learnt this so well that I could teach it to my peers.

Based on the data it was possible to interpret what kind of meanings the students constructed for the given mathematical expressions, and to evaluate how had they understood the mathematical concepts. The students were also asked to fill in a self-evaluation form to summarize their perception of their own knowledge of mathematical skills.

The students were aware of this study while data was collected during the course. They were able to choose whether their answers could be used in the study. All students gave permission to use their answers in the study. The students were informed that at all stages the processing of data is completely confidential and from the results of the study, the information provided by an individual student could not be identified. While students filled in a detailed self-evaluation form they used nicknames as the table was visible to all students. Students informed the teacher of their nickname. This research data will be used (in an anonymous manner) in this publications and in correspondence author's dissertation research/ when all the necessary data-based research has been done and then the data will be destroyed.

## 4 Results

This chapter presents the results of using languaging exercises on the course. First, the pre-test and post-test results are investigated for finding out how the students perceive their learning of the course topics. Second, the correlation between languaging skills and learning outcomes is investigated. Third, students' skills in different types of exercises (symbolic calculus, languaging and applied mathematics) are presented in relation to final grade. And finally, the self-evaluation form is used to analyze the students' perception of their own mathematical skills.

### 4.1 How does the students' languaging ability develop throughout the course?

On the course, the pre-test/post-test setup was used to see the possible gain in conceptual understanding, but these tests did not affect the final grade. The test was

exactly the same in the beginning and the end, and it consisted of six languaging exercises, where students explained in their own words the concept of derivative, interpreted a graph, and explained how the derivative of the given function is defined graphically, numerically, and symbolically. The exercises are shown in [Figure 7](#). [Figure 6](#) presents a word cloud of students' answers to open-ended question about derivative. For this figure the number of correct keywords in students' answers were analyzed.



Pre-test results		Post-test results	
			
Rate of change	7	Rate of change	35
Function	19	Function	21
Instant	0	Instant	16
Slope	9	Slope	14
Tangent	8	Tangent	13
Point	0	Point	9
Don't know/remember	16	Don't know/remember	1
(empty)	16	(empty)	3

Figure 6. Word cloud of how the students understood the concept of derivative (N = 53).

From [Figure 6](#) we can perceive that languaging through writing has improved during the course. The students are able to formulate the concept of derivative at the end of the course in a more versatile and correct way. Pre-test shows that at the beginning of the course the most common reply was that derivative is related to a function. There were many blank and Don't know/remember answers in the pre-test and only a few in the post-test.



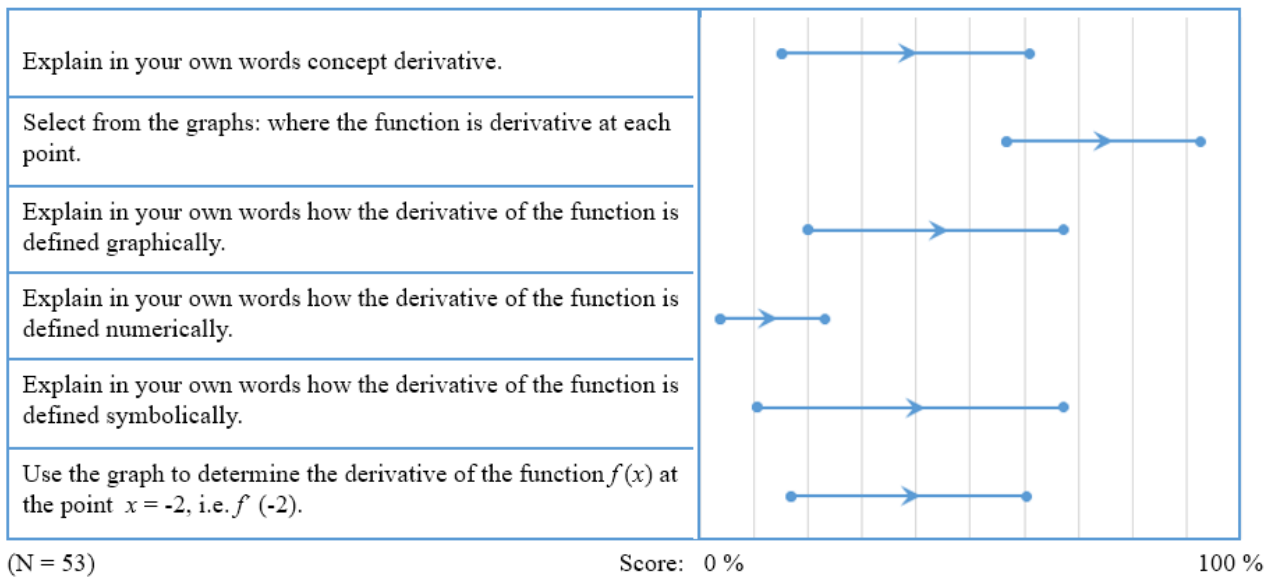


Figure 7. Gain chart.

In [Figure 7](#) are the averages of responses of 53 students to pre-test (left end of the line) and post-test (right end of the line). The left end of the line is the result of the responses to the exercise averaged over all respondents and the right end the responses to the final test, accordingly. Thus, the length of the line represents the average “amount of learning” during the course. It can be seen that explaining derivative symbolically, graphically, and the concept of derivative improved the most during the course. The exercise explaining in their own words the derivative graphically relates to the last exercise, where the students were actually asked to interpret the derivative of the function at the given point from a graph. In both exercises, the students improved very well during the course. Pre-test reveals that explaining in one’s own words what numerical derivative means was the least known matter and the learning outcome was also low here. The likely reason for this was that numerical solving was not practiced more than in a couple of exercises during the course. The second question had been answered well in pre-test and also in post-test. This question had three answer options, so this was a different type of question from the others, where the students had to explain in their own words or interpret the graph.

These results ([Figure 6](#) and [7](#)) indicates that the students’ ability to correctly express mathematical concepts by writing has improved. When students express their thoughts out loud and by writing, they remember things better and they are able to apply them later (Lee, [2006](#)).

## 4.2 How did the mathematical languaging clarify mathematical expression and how did develop the conceptual understanding?

Next, the correlation between the online exercises that were languaging exercises and the exam points (Figure 8) was calculated. Figure 8 shows the exam points ( $y$ ) as a function of points of languaging exercises ( $x$ ). The Pearson's correlation coefficient  $r = 0,68$  ( $N = 64$ ) tells a moderate positive linear correlation between the final assessment and languaging exercises. Exactly the same online languaging exercises were used on the Differential Calculus course during spring semesters 2018 and 2019. Also, the final exam on those years was delivered in a similar way with similar kinds of exercises and the correlation (Pearson's correlation coefficient) between the grading and languaging exercises was as follows: 2018  $r = 0,62$  ( $N = 58$ ) and 2019  $r = 0,62$  ( $N = 73$ ).

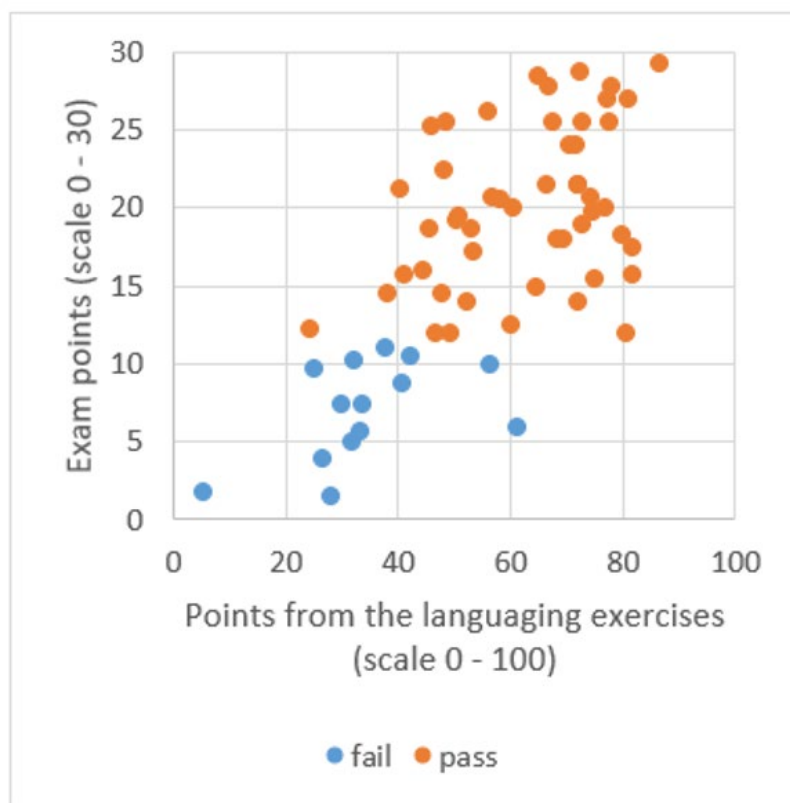


Figure 8. Correlation between the exam points and languaging exercises ( $N = 64$ ).

For a more in-depth study of how languaging exercises effects learning of concepts, the exercises in the final exam were investigated further (Figure 9). Figure 9 presents the average points of the exam exercises in different final grade categories from 0 (fail) to 5 (best). The blue line describes the average points in symbolic

calculations, orange line in languaging exercises, and grey line in applications. The orange line shows a clear step between grade 0 and 1 (increased 35 %). After this step, the curve shows only a minor increase in grade categories 1 - 5. Based on this shape of the curve, the competence of the languaging exercises has a clear effect on passing the course. This raises the question whether acquiring a certain level in languaging skills forms a threshold for understanding mathematics. To investigate more this very interesting finding, the current data was supplemented with data from two previous years. The languaging exercises were exactly the same every year and the exercises in the exams were similar. Even with this three times larger data set the result is the same: there is a clear step in the languaging category between the grade 0 and 1 (36 %, N = 195).

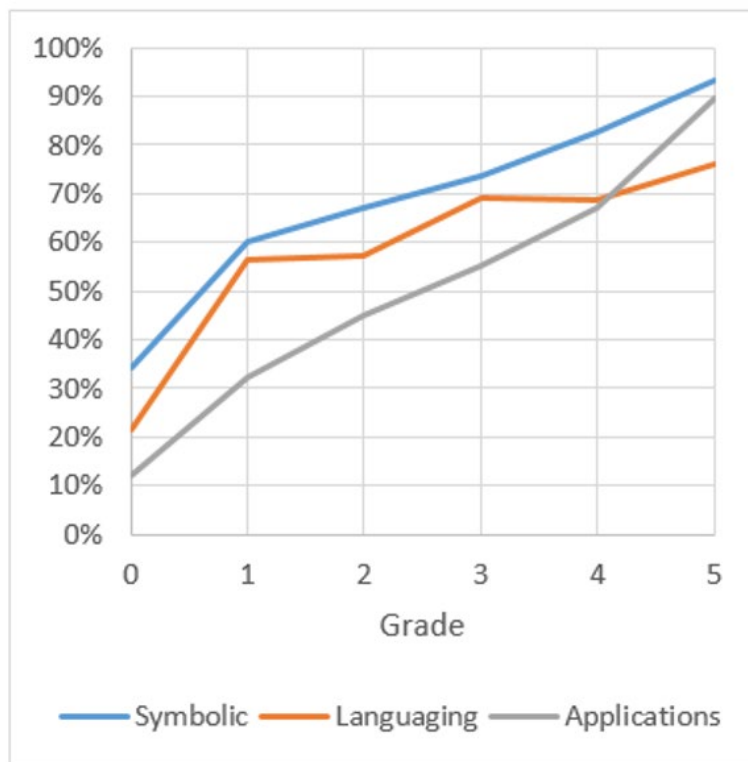


Figure 9. Exam exercises in three category (N = 64).

Research questions 2 and 3 (see page 8) dealt with the questions if mathematical languaging clarifies mathematical expressions and does that lead to the students' better conceptual understanding of the subject, which would help them to apply mathematics. Students, who did not pass the course, were able to do some mechanical calculations (symbolic) but did not get many points from the languaging exercises and even fewer from the applications (Figure 9). From Figure 9 we can also perceive that

students with a grade of five (5) stand out from other students in terms of competence in application exercises. They also got the best points in all categories. It seems that exercises where the students needed to apply their knowledge (grey line) has the highest discrimination power.

The students were also asked to fill in a detailed self-evaluation form weekly to collect their perception of their own knowledge of mathematical skills. The table was visible to all students. Therefore, only nicknames were used on the table. This table served many pedagogical purposes: it made the students evaluate their own knowledge about the key issues of the week, it made them think through languaging of the covered concepts, as the subjects of the week were explained by using mainly natural language, it showed them that others are perhaps struggling with the same topics as well and for the teacher, it showed which topics students had found the most difficult. The teacher then had the possibility to give extra guidance for the subjects that were found difficult.

Table 3 presents the averages of self-evaluations regarding the specific topic. The average of self-evaluations is calculated by first substituting the phrases (designated with letters a, b, c and d) with numbers 0 - 3 and then calculating the averages. Table 3 shows the same thing as Figure 9: the students estimated that they are good at doing mechanical calculations (symbolic calculations) and application tasks are more demanding.

**Table 3.** Averages of self-evaluations (scale from 0 (I need more practice to understand this) to 3 (I have learnt this so well that I could teach it to my peers.), N = 59

Topic	Average (and standard deviation) of self-evaluations
Regression and limit	2,15 (0,79)
Introduction to derivative (what is derivative - graphically, numerically and symbolically)	2,40 (0,64)
Symbolic calculations (derivative rules)	2,35 (0,77)
Applications (partial derivatives and error estimation)	2,11 (0,76)
Applications (finding maxima and minima using derivatives, Max-Min problems)	2,10 (0,83)
Applications (the derivative as a rate of change, tangent line, rates of change per unit time)	1,72 (0,69)

The Pearson's correlation coefficient between the final exam points and self-evaluation was a moderate positive linear correlation ( $r = 0,61$ ,  $N = 59$ ). The students

with higher grades (4 – 5) showed only slightly better self-evaluations compared with the group average. Students, who did not pass the course, seem to be overconfident of themselves, whereas the best ones seem to be somehow unsure of their knowledge and skills. Thus, the students seem to evaluate their knowhow “average”. Similar results have been found in another study of engineering students’ self-evaluations (Suhonen, 2019). This explains the rather small differences between averages in Table 3. Nevertheless, the Table 3 shows which topics are the most difficult ones to the students.

## 5 Discussion

In this study was presented the use of languaging method, in which the ways to express mathematical thinking are expanded beyond mathematic symbolic language. The objective was to observe how engineering students understand the concepts of differential calculus based on this method.

The use of languaging exercises on the mathematics course enables the teacher to interpret in more detail the students’ thinking and provides also a way for the teacher to evaluate the students’ understanding of the concepts. Also, the self-evaluation form provided the teacher valuable information of the difficulties the students encountered at the time while the subject was being covered, and thus the teacher was able to react and try to help the students proactively.

The analysis from the pre-test/post-test setup indicates that the students had learned expressing the meanings of the mathematical concepts by natural language. The findings also indicate that the students, who passed the course, were able to express mathematical concepts by natural language and to explain the meaning of the concepts.

Languaging exercises enable various types of ways to enhance the students’ mathematical thinking. Consequently, using different ways to express the mathematical concepts gives students a much clearer overall understanding of the mathematical concept in question. It seems that this helps especially those students, who struggle with mathematics (threshold of passing the course). Joutsenlahti and Kulju (2017) suggested that broadened ways of expressing mathematical thinking may help especially those students who have difficulties with mathematics and for whom mathematical symbolic language is difficult to comprehend.

The course Differential Calculus used various methods for learning mathematics alongside the languaging exercises, such as videos, visualizations, and learning

analytics. According to Moschkovich (2013), exercises that provide opportunities to participate in mathematical activities, which use multiple resources to do and learn mathematics support, among others, the mathematical reasoning and conceptual understanding. Kilpatrick, Swafford and Findell (2001) have pointed out that conceptual understanding is the ability to present mathematical solutions in different ways and the ability to evaluate how to utilize different presentations for different purposes. The understanding of mathematical concepts and the relationships between concepts will create sustainable development from the point of view of learning, which leads to the students being able to apply the mathematics later on in their engineering studies.

In the EDUCAUSE Horizon Report (2021), Horizon panelists were asked to describe key technologies and practices they believe will have a significant impact on the future of postsecondary teaching and learning. Six items rose to the top of a list as follows: Artificial Intelligence (AI), Blended and Hybrid Course Models, Learning Analytics, Microcredentialing, Open Educational Resources (OER) and Quality Online Learning. Three of these six technologies and practices identified in the report (learning analytics, OER, and AI) are returning entries from previous years' reports.

Learning analytics is a growing trend in all education. Many higher education institutions use digital learning management systems to deliver their courses, as was the case also in this study with the Differential Calculus course. These systems collect large sets of data about learners and their actions on the platform. Learning analytics, the data, offer a view, for example, to studying and learning activities, but learning management system cannot record such activities as reading the course book or carrying out calculations on paper. The data is a very valuable source of information to teachers, instructional designers, and the students themselves. However, it mostly tells about studying and does not reveal what has been actually learned and what is the student's perception of their own learning. In this study, the learning analytics view was combined with student self-evaluation, analysis data of online languaging exercises, and the actual learning outcomes. This forms a more comprehensive manner to look at the studying and learning and offers a way to try to find patterns and correlations. Further research is needed, but languaging exercises could be seen usable while creating education materials for learning and teaching mathematics, with the help of learning analytics and online learning to reveal mathematical thinking and conceptual understanding.



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# Identifying and promoting young students' early algebraic thinking

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Algebraic thinking is an important part of mathematical thinking, and researchers agree that it is beneficial to develop algebraic thinking from an early age. However, there are few examples of what can be taken as indicators of young students' algebraic thinking. The results contribute to filling that gap by analyzing and exemplifying young students' early algebraic thinking when reasoning about structural aspects of algebraic expressions during a collective and tool-mediated teaching situation. The article is based on data from a research project exploring how teaching aiming to promote young students' algebraic thinking can be designed. Along with teachers in grades 2, 3, and 4, the researchers planned and conducted research lessons in mathematics with a focus on argumentation and reasoning about algebraic expressions. The design of teaching situations and problems was inspired by Davydov's learning activity, and Toulmin's argumentation model was used when analyzing the students' algebraic thinking. Three indicators of early algebraic thinking were identified, all non-numerical. What can be taken as indicators of early algebraic thinking appear in very short, communicative micro-moments during the lessons. The results further show that the use of learning models as mediating tools and collective reflections on a collective workspace support young students' early algebraic thinking when reasoning about algebraic expressions.

Keywords: early algebraic thinking, learning activity, mathematical thinking, primary school, Toulmin's argumentation model

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## 1 Introduction

This article contributes an analysis of young students' potential to develop algebraic thinking. In a research review on mathematical thinking, Goos and Kaya (2020) point out that two broad aspects of mathematical thinking are mathematical problem-solving and mathematical reasoning. Translated to algebra, problem-solving and reasoning are activities in which indicators of algebraic thinking can be explored in the form of students' communicative actions. In this article, I will consider algebraic thinking as a part of mathematical thinking (e.g., Blanton et al., 2015; Cai & Knuth, 2011; Kaput, 2008; Kieran, 2004).

A basic condition for developing knowledge in different areas of mathematics is students' theoretical understanding of algebra (Davydov, 1990, 2008). Thus, algebra has a special position in mathematics since it is found in all other mathematical areas.



Both general reasoning in arithmetic, proof in number theory, and geometric formulas for area and volume use algebra as a working tool. In the form of equations, algebra is used in problem-solving in almost all mathematics. It is argued that a robust knowledge of algebra makes it easier for students to succeed in their further studies (Kieran et al., 2016; Matthews & Fuchs, 2020; see also Schoenfeld, 1995). However, many students find algebra difficult to learn (e.g., Carraher & Schliemann, 2007; Kieran, 2007; Matthews & Fuchs, 2020), and many teachers find it difficult to teach (Chick, 2009; Kilhamn et al., 2019; Røj-Lindberg, 2017; Røj-Lindberg et al., 2017). Algebra is also the mathematical area in which students perform poorly in all Nordic countries (Hemmi et al., 2021). Since it is regarded as challenging mathematical content, in the Western world algebra has tended to be introduced rather late, often as late as lower secondary school (see e.g., Bråting et al., 2019; Hemmi et al., 2021; Kilhamn & Røj-Lindberg, 2019; Stacey & Chick, 2004).

However, for several decades now there has been substantial interest in the youngest students' algebraic thinking, including their reasoning and problem-solving capabilities (e.g., Blanton et al., 2015; Bråting et al., 2019; Eriksson et al., 2019; Kaput, 2008; Lins & Kaput, 2004; Schmittau, 2004, 2005). Kaput et al. (2008) argue that it is not only possible but also beneficial to early develop students' algebraic thinking, in addition to their arithmetic thinking. In today's Nordic school mathematics curricula, Grades 1–6, algebraic content such as patterns, equalities, and equations are introduced (Børne- og Undervisningsministeriet, 2020; Skolverket, 2019; Utbildningsstyrelsen, 2020; Utdanningsdirektoratet, 2020; see also Bråting et al., 2019).

In the field of early algebra, reference is made to the so-called Davydov curriculum<sup>1</sup> in mathematics as a promising model for the development of algebraic thinking (Kaput et al., 2008; see also Cai & Knuth, 2011; Schmittau, 2004, 2005; Venenciano & Dougherty, 2014). This curriculum, with its roots in the Vygotsky tradition, is based on learning activity aimed at developing students' theoretical thinking in mathematics, and foremost their algebraic thinking (Davydov, 1990, 2008; Schmittau, 2004, 2005). Central to the Davydov curriculum is mediating tools, what he calls learning models, and collective reflections, which are used as a means for supporting students' theoretical work (Davydov, 2008; Gorbov & Chudinova, 2000; Repkin, 2003; Zuckerman, 2003).

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<sup>1</sup> *The Davydov curriculum* is also referred to as *Davydov's programme*, and the *El'konin-Davydov curriculum (ED curriculum)*.

Less is known about how young students' algebraic thinking emerges during classroom work and how it can be identified (Goos & Kaya, 2020). There is thus a need for empirical examples of how early algebraic thinking among young students can be identified and promoted, and what in the design of tasks, tools, and communicative resources has the potential to enhance students' early algebraic thinking (e.g., Goos & Kaya, 2020). Given that thinking cannot be analyzed as such (Radford, 2008a, 2010), there is a need to use, for example, the students' tool-mediated communicative actions as indications of their algebraic thinking.

## 1.1 Aim and research questions

In this article I analyze young students' communicative actions on algebraic expressions<sup>2</sup> to identify indicators of early algebraic thinking and discuss what in a learning activity promotes students' opportunities to explore algebraic expressions. The aim is specified in two research questions (RQs):

- RQ1      What in young students' tool-mediated communicative reasoning can be taken as indicators of their early algebraic thinking?
- RQ2      What in the learning activity promotes young students' early algebraic thinking when exploring algebraic expressions?

## 2 Background and research on algebraic thinking

Several researchers in the field of early algebra argue that challenges concerning algebraic thinking may be due to algebra usually being introduced through arithmetic, for example in the form of tasks focusing on equalities in which the value of an unknown number is requested (e.g., Kieran, 2006; Kieran et al., 2016; see also Lins & Kaput, 2004; Stacey & MacGregor, 1999). This may be a reason that difficulties arise later in algebra learning, where students often get stuck on numerical solutions (see e.g., Kaput, 2008; Kieran, 2006; Radford, 2010). Therefore, it is argued that the introduction of algebra should promote algebraic thinking from the beginning of primary school (Lins & Kaput 2004; Roth & Radford, 2011). Kieran (2004) suggests that students need to work theoretically in different ways at an early stage. Thus, they

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<sup>2</sup> In this article, an algebraic expression refers to a meaningful composition of mathematical symbols (Kiselman & Mouwitz, 2008). This implies, for example, that  $x + y - z$  and  $yx + z$ , but also the inequality  $x < y$  and the equality (or equation)  $x = y + z$ , are expressions (James & James, 1976). In the study on which this article is based, we have used algebraic expressions in the form of equalities of the type  $a = b + c$ .



need to encounter tasks and problems that promote algebraic thinking; for example, they need to have the opportunity to explore algebraic structures (Blanton et al., 2015; Bråting et al., 2018; Kieran et al., 2016).

To promote the development of algebraic thinking, teachers need to create conditions for students to develop abilities such as reasoning algebraically, making algebraic generalizations, and using algebraic representations, rather than teaching several procedures (Greer, 2008; Kaput, 1999; Usiskin, 1988). Mediating tools can furthermore play a crucial role in developing a so-called “non-counting” approach, whereby students work with problem-solving tasks (Schmittau, 2004; Venenciano & Dougherty, 2014). Algebraic thinking can involve theoretical work with letter symbols or other relational resources in the students’ analysis of, for example, relations between quantities, structures, and patterns. This also includes working with justifying and proving (Kieran, 2004).

Kaput (2008, p. 11) highlights two core aspects that account for algebraic thinking: “(A) Algebra as systematically symbolizing generalizations of regularities and constraints. (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems”. Thus, the development of algebraic thinking includes, among other things, the exploration of general, fundamental, and theoretical relationships and structures (see also Blanton et al., 2015; Davydov, 1990, 2008; Venenciano & Dougherty, 2014).

Matthews and Fuchs (2020) point out the relational aspect in the equal sign as an especially important component of algebraic thinking, referring to it as a “big idea” in mathematics (Matthews & Fuchs, 2020, p. e15). Instead of understanding the general structure, students often understand it as an operator that implies a sum or a result. Thus, the students need to develop a relational view of the equal sign and interpret it as “the same as” (Matthews & Fuchs, 2020, p. e15; see also MacGregor & Stacey, 1997; Warren & Cooper, 2009). Schmittau and Morris (2004) state that it is possible for young students, by comparing quantities, to theoretically work with inequalities and equalities. When children write

“If  $C < P$  by  $B$ , then  $C = P - B$  and  $C + B = P$ ”; the notation indicates that they can move from an inequality to an equality relationship by adding or subtracting the difference, and that addition and subtraction are related actions. (Schmittau & Morris, 2004, p. 81)

Also, Blanton et al. (2015) point out the relational understanding of the equal sign as important and include this in the big idea of *equivalence, expressions, equations, and*



*inequalities* (EEEI). Further, Blanton et al. (2015, p. 43), as part of EEEI, include “representing and reasoning with expressions and equations in their symbolic form and describing relationships between and among generalized quantities that may or may not be equivalent”.

Ventura et al. (2021) argue that young students can use the function of variables in algebraic expressions (see also Venenciano et al., 2020). In a Nordic context, however, few studies have explored algebraic thinking with symbols and relational material. Eriksson et al. (2019) used non-numerical examples, in the form of  $a = b + c$ , when introducing algebraic expressions in grades 1 and 5 (see also Eriksson & Jansson, 2017; Wettergren et al., 2021). Given that algebraic expressions are central aspects of algebra, the use and understanding of indeterminate quantities is considered crucial for the development of algebraic thinking (Ventura et al., 2021). In the study on which this article is based, the tasks have been constructed in line with Kaput’s (2008, p. 13) description: “the initial symbolization uses letters to denote quantities, thereby embodying generality in the symbolic expression of specific (but unmeasured) cases involving, say, comparisons of lengths”.

### 3 Learning activity as a theoretical framework

According to Vygotsky (1986), a prerequisite for developing theoretical thinking is a teaching in which children are allowed to encounter scientific (theoretical) concepts at an early age, compared to being introduced to everyday (empirical) concepts. With reference to Vygotsky, Schmittau (2004, p. 39) argues that “in order to learn mathematics as a *conceptual system*, it is necessary to develop the ability to think *theoretically*”. Thus, students need to develop theoretical thinking early, through a teaching that offers them opportunities to engage in work with concepts and their relations and structures.

The Davydov curriculum in mathematics and learning activity is based on the idea of “ascending from the abstract to the concrete” (Davydov, 2008, p. 106). He argues that students first need to work with general structures and relations in, for example, algebraic expressions in order to later use them in concrete numerical operations. A basic principle of learning activity is that theoretical thinking related to mathematical concepts needs to be explored with the help of mediating tools, in learning activity referred to as *learning models* (Davydov, 2008; Gorbov & Chudinova, 2000; Repkin, 2003). Learning models can be seen as materialized representations of the abstract (Repkin, 2003). These can be constructed of physical representations (e.g., Cuisenaire

rods<sup>3</sup>), symbols (e.g., in the form of variables), schemes represented with lengths (I—I), and graphs (e.g., in coordinate systems). According to Davydov, learning models have different functions that aim to enable students to theoretically explore the abstract (structural) aspects of a given object of knowledge (Davydov, 1990, 2008; Davydov & Rubstov, 2018; Gorbov & Chudinova, 2000). Davydov emphasizes that “not just any representation can be called a learning model, but only one that specifically fixates the universal relation of some holistic object, enabling its further analysis” (Davydov, 2008, p. 126). The intention of a learning model is to make certain aspects of an object visible, and it functions as a tool when students work on a problem. A learning model can also function as a tool for classroom communication (Davydov, 2008). Radford (2008b, p. 219) argues that the tools “are not merely aids: their mediating role is such that they *orient* and *materialize* thinking and, in so doing, become an *integral* part of it”. In other words, learning models can visualize students’ thinking and thus constitute a mediating tool in the work with concepts.

Another basic principle in learning activity is that students be given the opportunity to participate and engage in *collective reflections* (Zuckerman, 2003, 2004). Thus, the mathematical content can be made visible, explored, and developed as a conceptual understanding. Students’ experienced motive for engaging in theoretical work can be made possible when groups of individuals are allowed to work together and share or borrow each other’s experiences and knowledge (Vygotsky, 1986; Zuckerman, 2004). Thus, reflection is not seen as an individual process but rather takes place collectively. The starting point for collective reflections is that students, by engaging with other students’ suggestions and explanations, can understand their own thinking (Zuckerman, 2003).

To realize a learning activity, for example, regarding algebraic expressions, the teacher has to enable and pursue an elaborative discussion among the students (Zuckerman, 2004). When planning for a learning activity, the teacher’s choice of a task framed as a problem situation, as well as considerations of how the structural aspects can be visualized, are crucial (Repkin, 2003). Thus, students should encounter a problem situation that requires work and that can result in the development of their theoretical thinking. Such a problem situation must be perceived by the students as meaningful; that is, they should experience a need to explore the problem. The students’ exploration of the problem situation comes about through collective reflections together with the teacher and can take the form of class discussions. These

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<sup>3</sup> Cuisenaire rods are a relational laboratory material that consists of rods of different lengths and colors, with each length being a certain color (Küchemann, 2019).

can take place on what can be understood as a collective workspace, for example when the representations and symbols are displayed on a whiteboard, which has a decisive function in this respect (Eriksson et al., 2019).

## 4 Method

This article is based on a study which took the form of a design experiment (e.g., diSessa & Cobb, 2004). In this study, we, four researchers and five teachers worked collaboratively. That is, the researchers and teachers iteratively planned, adjusted, and refined the lessons and problem situations together (Carlgren, 2012; Eriksson, 2018). The study's focus was on developing mathematics teaching; more precisely, on promoting young students' early algebraic thinking and their reasoning about algebraic expressions through a collective and tool-mediated teaching situation.

### 4.1 Data

The study was conducted at two municipal schools, with 550 and 1150 students respectively, and from preschool class to Grade 9. The five participating teachers had between 15 and 23 years of teaching experience. The all signed up to the project voluntarily. Four of the teachers had a Grade 1–6 teaching qualification and one had a Grade 4–9 teaching qualification. Forty-two students across grades 2 to 4 participated in the various research lessons (Table 1).<sup>4</sup>

**Table 1.** Research lessons conducted 2015–2017

School year	Grade	Students in research lesson 1	Students in research lesson 2	Students in research lesson 3	Total students in the research lessons
2015–2016	2 <sup>5</sup>	6 <sup>6</sup>	6		12
2016–2017	3	6	6	6	18
2016–2017	4	4	4	4	12

Inspired by learning study (Marton, 2005, 2015; Runesson, 2017), eight research lessons were conducted in grades 2, 3, and 4 (students aged 7–10) at two schools during the period 2015–2017. One research lesson cycle in each grade was held with

<sup>4</sup> The research lessons were conducted within the context of the mathematics network at *Stockholm Teaching & Learning Studies* (STLS).

<sup>5</sup> Only two research lessons were carried out in Grade 2

<sup>6</sup> Research lesson 1 in Grade 2 has been excluded due to administrative complications.

different groups of students in the same grade. The cycles were conducted iteratively over the school year. The research lessons, lasting between 26 and 44 minutes, were video recorded. Altogether, the lessons amounted to 251 minutes. None of the student groups had previously worked with algebraic expressions or formal equations during the school year in which the data were collected. The data in this article are taken from the second research lesson in each grade since the students' collective reasoning on the general structures and relational aspects of the algebraic expressions was rich.

The jointly planned lessons, all in the form of whole-class discussions, were conducted by one teacher in each grade. The other teachers in attendance were responsible for the data collection, such as video recording and observation. As the focus of the recording was the joint activities on the whiteboard in front of the class, the video camera was mostly pointed at the whiteboard. The research lessons were transcribed in their entirety according to Linell's (1994) description: word-for-word, speech-neutral text, organized in dialogic form.<sup>7</sup> Interaction in the form of gestures and concrete manipulations, when these appeared in the video, are also described in brackets in the excerpts since they can be seen as part of the argumentation (Nordin & Boistrup, 2018). The students were given fictitious names.

## 4.2 The design of the research lessons

The design of the research lessons was inspired by learning activity (Davydov, 2008) and the previously mentioned concepts of learning models (Repkin, 2003) and collective reflections (Zuckerman, 2003). In addition, special attention was paid to enabling joint work on the collective workspace (Eriksson et al., 2019; cf., Liljedahl, 2016); that is, on the classroom whiteboard.

The overarching aim of each lesson was for the students to discern the relations between quantities, structures, and general patterns in algebraic expressions. Therefore, the problem situations they were to explore and reason in relation to consisted of contrasting examples of visualizations of algebraic expressions. Having the students encounter problem situations with alternative solutions and asking them to reflect on and explain someone else's solution made it possible for them to take another person's perspective (Zuckerman, 2004). Also, in the design of the teaching, learning models were used as mediating tools. These took the form of Cuisenaire rods

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<sup>7</sup> As the research lessons were conducted in Swedish, the transcripts have been translated into English.

(Küchemann, 2019).<sup>8</sup> Line segments drawn on the whiteboard and symbols for variables were also used as learning models. To promote students' algebraic thinking, the teachers planned the possible types of responses they could employ depending on the situation and took into account that students' discussions could either stop or take a less desirable direction than had been hoped. Considering that students' participation in a learning activity should be characterized by their agency, how the teacher responds is important. For example, asking "how did xx think here?" instead of addressing the individual student and asking, "how did you think here?" can lead to different results.

The overall structure of the research lessons was the same. The lessons started with the presentation of a learning model in the form of line segments or Cuisenaire rods visualizing a relation that was to be collectively reflected on. The teacher worked with the students at the whiteboard. Also, each student had access to tools in the form of rods on the table in front of them. The teacher was responsible for maintaining the students' collective reasoning through questions and provocations. Occasionally, the students approached the whiteboard when they were to present their suggestion or solution to the given problem situation.

### 4.3 Analysis

Toulmin's model of argumentation (Toulmin, 2003) was used to analyze the class discussions for possible indications of algebraic thinking. Toulmin's model is a theoretical model of an argument and has most commonly been used in research on interaction within mathematics education, for example, proof (e.g., Hemmi et al., 2013), often in the reduced version introduced by Krummheuer (1995). The reduced model consists of four elements, three of which—claim, data, and warrant—are regarded as the core of an argument, along with a potentially fourth element, backing. A claim is a statement that is grounded in data, and the warrant functions as a bridge between data and claim. According to Toulmin (2003), the data supporting the claim can answer the question "What have you got to go on?" (Toulmin, 2003, p. 25) and the warrant would answer "How do you know?" (Toulmin, 2003, p. 210). In an analysis of the interaction, the argument and each of the four elements can be created by more than one individual. The elements do not need to be expressed in a specific order and can be expressed in many ways, for example verbally, with written symbols,

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<sup>8</sup> The Cuisenaire rods used in the study were comprised of various materials, some wooden and some magnetic, the latter designed to be used on a whiteboard.

drawings, or gestures, or using manipulatives, in this case Cuisenaire rods (Nordin & Boistrup, 2018).

The analysis began with a reading of the transcripts for the research lessons in each of grades 2, 3, and 4 in their entirety. In each transcript, sequences were identified. Here, a sequence is the time between the teacher's presentation of a new problem situation with an algebraic expression or a learning model visualizing an algebraic expression, and the presentation of the next problem situation. Within each sequence, I searched for various reasonings that used arguments with claims and highlighted them. I then searched for data supporting each claim. If I found data supporting the claim, I searched for the warrant. In some cases, the students also gave non-mathematical suggestions; for example, the variables needed to be in alphabetical order, or a specific value had to be requested. Therefore, an additional delimitation of an argument concerning algebraic thinking was that it should focus on the relational and general aspects. The initial analysis was also discussed with the participating teachers, a process described by Wahlström et al. (1997) as *negotiated consensus*. All video recordings and transcripts were reviewed.

When the elements of an argument were identified, a reconstructed argument was written. To clarify the reconstructed argument in the excerpts below, I have written the elements claim, data, and warrant in brackets. I present the excerpt, followed by a reconstructed argument following Toulmin's reduced model. After the reconstructed argument, I interpret the indicator of early algebraic thinking.

Reconstructing the arguments indicated early algebraic thinking in the students' arguments. Following Radford's (2008a) idea that students' communicative actions can be understood as a form of reflections of thinking, aspects of the students' theoretical work were identified. However, while students' communicative actions are not to be equated with theoretical thinking, they can serve as indicators of theoretical thinking, in this case early algebraic thinking.

## 5 Results

I present the results from the analysis of the research lessons below. Three empirical examples, not age-specific, are chosen to exemplify indications of students' early algebraic thinking, one each from grades 2, 3, and 4. The chosen examples represent the focus in the class discussion in each grade.



## 5.1 Grade 2: Establishing equalities

In Grade 2, the teacher introduced algebraic expressions by presenting a make-believe café where the students were to work. This café sold goods such as buns, chocolate bars, and sandwiches. Initially, the students were to construct a price list for the goods. There was no ordinary money; the prices of the goods had to be represented by something else and the teacher suggested Cuisenaire rods. The teacher said: “in this café that you work in, a bun costs a purple rod, like this [holding up a purple rod] ... You do not have regular money here; we only have rods like this.” The rods were on a table in front of the students, and the teacher had placed corresponding rods in the form of magnetic strips on the whiteboard. In establishing the price list, the teacher said that a bun cost one purple rod (Figure 1).



Figure 1. Two buns cost four red rods.<sup>9</sup>

### Excerpt 1 (60 seconds)

- Teacher: So, if you were to come to me and buy a bun, what would you pay then? Karin, what would you pay?
- Karin: A purple [rod]. **[data]**
- Teacher: A purple [rod]. And Seydou [the teacher points to the student], what do you think? What would you pay for a bun?
- Seydou: Are there also others [referring to the rods on the whiteboard]?
- Teacher: How do you mean?
- Seydou: Could it also be that money [referring to other rods on the whiteboard] ... with some others too?
- Teacher: The bun costs just that [points to the horizontal purple rod]. **[data]** But did you want to pay with something else?
- Seydou: Yes.
- Teacher: Some other rods like this? [points to all the rods on the whiteboard]. Which [rod] would you like to pay with, then?
- Seydou: Four red [rods]. **[claim]**

<sup>9</sup> “Bulle” and “Chokladkaka” are the Swedish words for “bun” and “chocolate bar”, respectively.

- Teacher: Four such [places four red rods, horizontally, centered under the purple rod to the right of the word "bun"]. How did you think now, can ... why do you want to pay with four red [rods]?
- Seydou: Because look, two are one ... [in his intonation indicating that two red rods correspond to one purple rod] **[warrant]** ... I wanted to buy two buns [with an indication of emphasis in his tone]. **[claim]**

First, Seydou indicates an understanding that a bun which costs one purple rod corresponds with two red rods. Second, in his argument he elaborates with the new value and currency of the bun (two red rods) when he argues that he intends to buy two buns and therefore wants to pay with four red rods. With his claim and warrant, Seydou demonstrates an understanding of the principle of equality. In the following, the reconstructed argument for why two buns cost four red rods is presented (Figure 2).

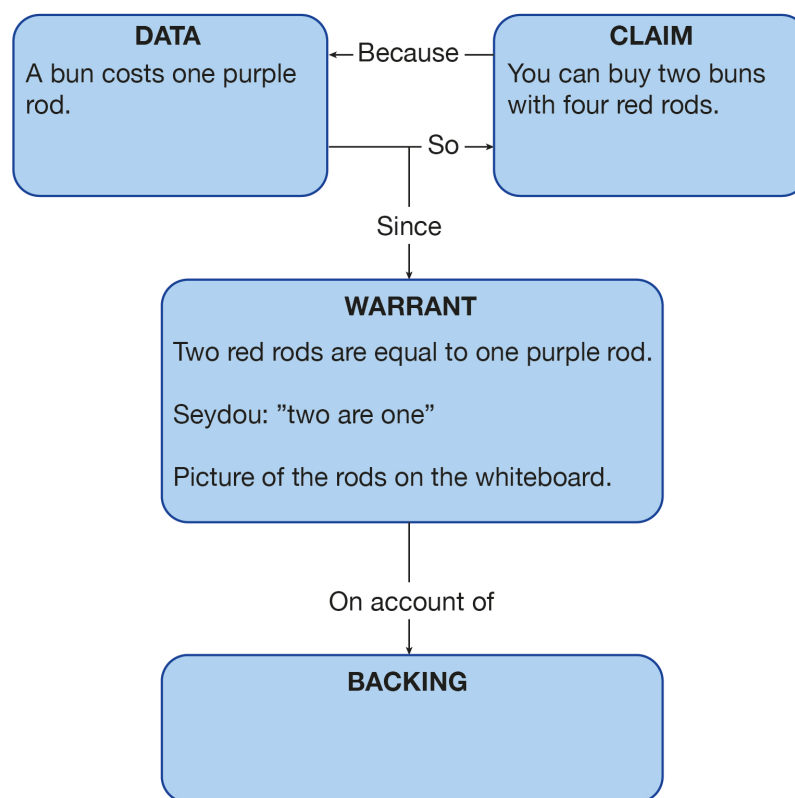


Figure 2. Reconstructed argument for why two buns cost four red rods.

The reconstructed argument shows how Seydou implicitly uses the learning models when pointing out the relation between the different rods, with their lengths set to have a non-numerical value, by choosing rods that equal the purple one. The indicator of early algebraic thinking in this case involves establishing equalities.

## 5.2 Grade 3: Adjusting inequalities to equalities

In the Grade 3 research lesson, the group had previously worked on a problem situation in which they created equalities that would correspond to the expression  $z = x + y$ . This was done using learning models where the students worked with rods on the tables and the teacher drew corresponding vertical line segments on the whiteboard. Based on the student examples, several alternatives were created, all of which corresponded to the algebraic expression. In connection with this, the group had a shorter discussion where it was stated, among other things, that the constructions needed to have the same length **[data]**. The teacher then drew new line segments on the whiteboard, with the one on the left representing  $z$  and on the right a combined line segment representing  $x$  and  $y$  (Figure 3).

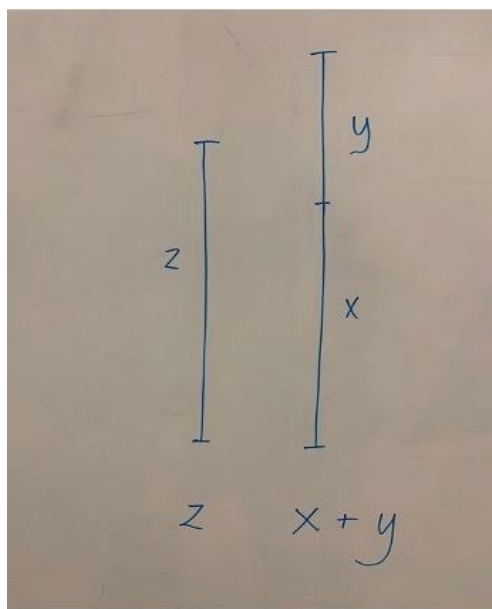


Figure 3. Constructed model of line segment  $z$  and line segment  $x$  and  $y$ .

This constructed a learning model in which the line segments were no longer equal **[data]**, even though they related to the same algebraic expression as before ( $z = x + y$ ). The students collectively expressed that this was not correct and came up with some suggestions for how to make the line segments and the expression match. One of the students, Sisay, was given the floor:

Excerpt 2 (1 minute 16 seconds)

- Sisay: You could also ... above  $z$ , draw a small “block” [referring to a small rod shaped like a cube on the table in front of her] that we name  $w$  or something, eh ... **[starting a claim]**
- Teacher: [Measures with thumb and index finger as a distance above  $z$  so the line segment on the left will be as long as those on the right] so that they become equal [draws a new line segment, which she calls  $w$  so that the left now consists of two line segments ( $z$  and  $w$ )] **[warrant]**
- Sisay: Because then ... in that case it will be  $z$  plus  $w$  is equal to  $x$  plus  $y$ . **[claim]**
- Teacher: Like this,  $z$  plus  $w$  is equal to  $x$  plus  $y$  [writes  $z + w = x + y$  below the line segments on the whiteboard] (see [Figure 4](#)). Mm. What does Elsa think of this?
- Elsa: Eh ... yes.
- Teacher: Or do you think, what do you think? Do you agree ... do you understand what Sisay means? Do you think that one can do this?
- Elsa: Mm.
- Teacher: Why can you do that, then? You know, yes ... but this is what we think we can do, but what is it that makes us think we can do it like that? What makes it feel like it's right when you do it like that? Lisa.
- Lisa: That they'll have the same length.  $x$  plus ... **[warrant]**
- Teacher: They're the same length [points to  $w$  and  $z$  in the line segments to the left].
- Lisa: Yes,  $x$  plus  $y$  and  $z$  plus  $w$  will be equal in length. **[warrant]**

An indication of early algebraic thinking is when Sisay adjusts the learning models ([Figure 4](#)), that is, the line segments on the whiteboard, from an inequality to an equality. In doing so she adds a new symbol, which she decides to name  $w$ .

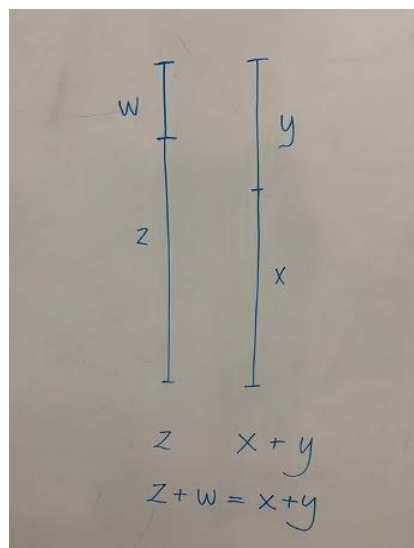


Figure 4. The expression  $z + w = x + y$  and adjusted model for the corresponding expression.

Thus, the line segments need to have the same length in order to be equal. When adding a new length and naming it “ $w$ ” the line segments become equal. However, it is not enough to only adjust the line segments, also the expression needs to be complemented to  $z + w = x + y$  to correspond. In the following, the reconstructed argument for why you need to add “ $w$ ” is presented (Figure 5).

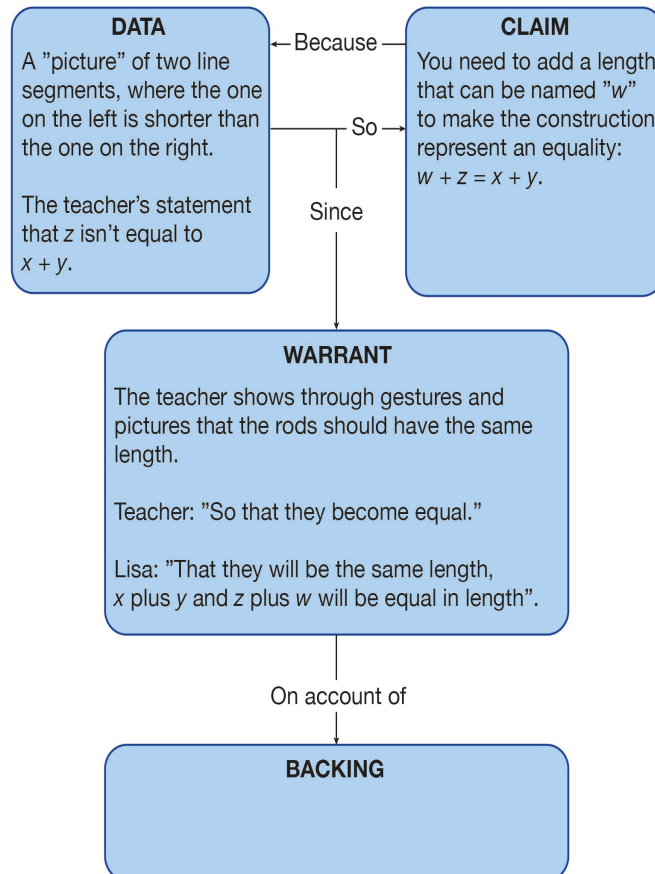


Figure 5. Reconstructed argument for why you need to add “ $w$ ”.

The reconstructed argument visualizes Sisay's and Lisa's respective understandings of both the equal sign and the functions of variables when adjusting inequalities to equalities.

### 5.3 Grade 4: Generalizing equalities

During the lesson in Grade 4, the group worked collectively on the algebraic expression  $a = b + c$ . The teacher wrote the expression on the whiteboard [**data**], after which the students constructed variations of the expression with the rods, as a learning model, on a table. The students then were asked to display their constructions on the whiteboard (Figure 6).

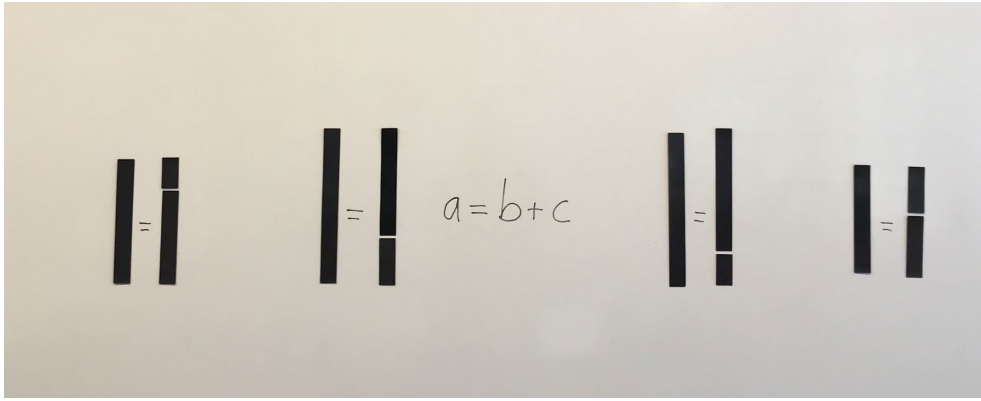


Figure 6. The students' constructions of the expression  $a = b + c$ .

The first construction to the left and the fourth construction to the right used rods of equal length to represent  $a$ , while the second and third constructions in the middle of the whiteboard used rods longer than those in the former constructions (also of equal length) to represent their  $a$ . The lengths of the rods representing  $b$  and  $c$  varied in all constructions. Thus, four different representations of the equality  $a = b + c$  appeared **[data]**, all of which were valid representations of the algebraic expression  $a = b + c$  **[warrant]**. After a short discussion in which the students expressed that all the alternatives were correct, the teacher asked why  $a$  might differ.

Excerpt 3 (50 seconds)

- Teacher: Mm, why might they, the  $a$ 's, differ [with reference to the fact that there are different large  $a$ 's in the rod constructions on the whiteboard]? **[data]**
- Johan: Eh ...
- Teacher: Okay, Johan.
- Johan: It depends a bit on what we think  $a$  can be ... because we should still build  $a$  ... that  $a$  ... describe with these, [with reference to the rods on the whiteboard] **[warrant]** that  $a$  is equal to  $b$  plus  $c$  **[claim]**, [referring to earlier discussion] so I suppose we could choose a template ... kind of like this length [refers to a rod he lays in front of him on the table] and then we take  $b$  ... in other words, another [puts a rod next to the newly laid rod] ... plus ... we can take something completely different here ... then we choose  $b$  plus  $c$  [starts adding a third rod above the last laid rod] ... sorry, now I have the wrong rod [referring to the rods he has on the table which result in an inequality], but if you think about mine over there [points to his rod construction on the whiteboard], one ... one  $b$  plus  $c$  ... one ... so we kind of made a new " $a$ " of two other pieces. **[claim]**



Indication of early algebraic thinking is the reasoning about general structures in Johan's utterances: "It depends a bit on what we think a can be," "I suppose we could choose a template," and "we can take something completely different here." When there are not enough rods on the table, he chooses to relate to his own construction on the whiteboard after initially starting with a randomly selected rod. In the following, the reconstructed argument for why  $a$  can differ in length and still have the value of  $a$  is presented (Figure 7).

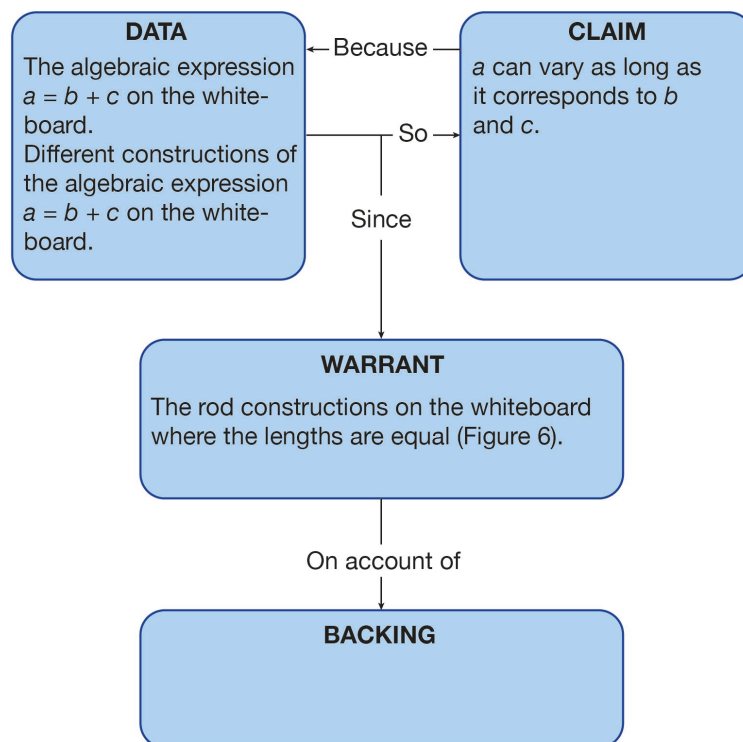


Figure 7. Reconstructed argument for why  $a$  can differ in length and still have the value of  $a$ .

The reconstructed argument above shows that the same variable can have different lengths. That is, the student apparently does not need to decide the value of  $b$  and  $c$  as he reasons about the algebraic expression without determining the value of the variables. The indicator of early algebraic thinking in this case involves generalizing equalities.

## 6 Discussion

As presented above, the results give a set of indicators of early algebraic thinking among young students, empirically exemplified in the excerpts. The results also exemplify how early algebraic thinking can be identified. It could be argued that the

small number of students in each group in the study does not represent a realistic teaching situation, where the student groups are typically much larger. However, the communicative actions analyzed in this study contribute to knowledge about identified indicators of early algebraic thinking among young students in micro-moments, which can be overlooked in daily teaching in whole-class settings or situations. The results also point to the importance of planning the teaching situation (that is, the problem situation, learning models, and the teacher's responses) to engage the students in an explorative algebraic learning activity.

In the next sections, I discuss the results more closely in relation to the aim of the article and earlier research.

### 6.1 Indicators of early algebraic thinking

In line with research on early algebra, the results confirm an emergence of young students' early algebraic thinking when working with general structures and relationships in algebraic expressions (Blanton et al., 2015; Bråting et al., 2018; Kaput, 2008; Kieran, 2006; Kieran et al., 2016). Above all, two of the three core concepts of students' early algebraic thinking that Ventura et al. (2021, p. 4) highlight were found: "the relational understanding of the equal sign" and "generalizing and representing indeterminate quantities in algebraic expressions".

However, as the results show, it was only during short moments that an indicator of early algebraic thinking could be identified. The indicators identified in this study are: 1) establishing equalities, 2) adjusting inequalities to equalities, and 3) generalizing equalities. The student in Grade 2, while elaborating on/with the learning models, was able to establish an equality. In doing so, Seydou expressed an understanding of equality when he argued that two buns could be bought with four red Cuisenaire rods. An understanding of the equal sign is something that Matthews and Fuchs (2020) mention as important for students to interpret. Besides exhibiting the important ability to interpret "the same as" (Matthews & Fuchs, 2020, p. e15), the students in Grade 3 also adjusted an inequality given in the learning models by adding a length and naming it with the symbol "w". Also, the expression  $z = x + y$  was adjusted to  $z + w = x + y$ . In doing so, they moved from an inequality to an equality, which according to Schmittau and Morris (2004) is an example of algebraic thinking (see also, Eriksson & Jansson, 2017; Kieran et al., 2016). Johan, in Grade 4, showed indications of Kaput's (2008) core aspects; that is, when generalizing on the equalities he reasoned algebraically. Kieran (2004) highlights justifying and proving as

examples of actions that involve algebraic thinking. In addition, the result indicates that the students collectively developed their understanding of the concept of variables through the learning models. While they did not use the term variable, they demonstrated their understanding through everyday language and gestures.

## 6.2 Promoting early algebraic thinking

The collective reflections in the research lessons were made possible through the joint tool-mediated theoretical work on which the reasoning was based. The learning models visualize, or as Radford (2008b, p. 219) argues “*orient* and *materialize*” the students’ theoretical thinking on the relational structure in the algebraic expressions. Further, the learning models on the collective workspace visible to everyone contributed to these reflections (Eriksson et al., 2019). In the collective reflections, the teachers’ responses were crucial in establishing and maintaining the learning activity. For example, the teachers were not content when a student gave a correct answer. Instead, they questioned the student’s answer by saying “I don’t understand,” “[c]an this really be true?” or asking another student to explain the given solution. Also, the collective reflections made it possible for the students to, so to speak, borrow knowledge from each other which enabled them to qualify their reasoning.

In the Grade 2 lesson, the teacher used the student’s suggestion to challenge him to engage in the theoretical work. That is, the teacher’s question “[w]hich [rod] would you like to pay with, then?” required a claim that needed to be substantiated. In Grade 3, the teacher created an example in which the learning model did not correspond to the given algebraic expression. This required that students explore the problem situation, and they collectively manipulated the line segments to create an equality, not by extending the existing line segment named  $z$  but by adding a new one that Sisay decided to call  $w$ . In Grade 4, the teacher’s question along with the students’ rod constructions on the whiteboard challenged the students and promoted algebraic thinking. Although all the different examples corresponded to the algebraic expression, the teacher was not “satisfied” with/did not settle for this. Instead, by departing from the students’ different constructions, the teacher prompted them to argue for how the different equalities in the rod constructions could all correspond to the same algebraic expression.

## 6.3 Concluding remarks

Algebraic thinking is a part of mathematical thinking, and the results illustrate how

collaborative tool-mediated reflections can promote the development of students' early algebraic thinking. It should be noted that, according to the teachers, the participating students had not worked with this type of algebraic expression previously during the current school year. Moreover, aspects of the teaching situation were new to them. An example of the new teaching situation was that the teacher did not directly confirm whether the students' suggestions were correct. Another example was the problem situations upon which the students were expected to elaborate, since they all consisted of non-numerical but tool-mediated examples. The students were also unaccustomed to collectively working at the whiteboard and elaborating on the theoretical content in front of the student group (Zuckerman, 2004) as well as to working with learning models (Repkin, 2003). The establishment of a learning activity in which students can experience a motive, create a learning task, and collectively explore the theoretical content has the potential for developing students' relational agency (Edwards, 2005). However, learning activity is fragile, so to speak, and whether students establish a learning activity (Davydov, 2008) depends on several factors. For example, the subject-specific teaching situation in the form of a problem situation needs to highlight theoretical content and be designed to enable the students to perceive that there is a real problem to solve (Repkin, 2003). Furthermore, teaching based on the central principles of learning activity differs from much of the mathematics teaching in Swedish classrooms (Bråting et al., 2019; Hansson, 2011; Johansson, 2006; Larsson & Ryve, 2012).

These results allow reflection on what can promote and enhance young students' early algebraic thinking and the identified indicators exemplify what teachers can pay attention to when striving to develop students' algebraic thinking when working on algebraic expressions.

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# “Learning models”: Utilising young students’ algebraic thinking about equations

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The overarching aim of this article is to exemplify and analyse how some algebraic aspects of equations can be theoretically explored and reflected upon by young students in collaboration with their teacher. The article is based upon an empirical example from a case study in a grade 1 in a primary school. The chosen lesson is framed by the El’konin-Davydov curriculum (ED Curriculum) and learning activity theory in which the concept of a *learning model* is crucial. Of the 23 participating students, 12 were girls and 11 boys, approximately seven to eight years old. The analysis of data focuses on the use of learning models and reflective elaboration and discussions exploring algebraic structures of whole and parts. The findings indicate that it is possible to promote the youngest students’ algebraic understanding of equations through the collective and reflective use of learning models, and we conclude that the students had opportunity to develop algebraic thinking about equations as a result of their participation in the learning activity.

Keywords: The El’konin-Davydov Curriculum, learning activity, learning models, algebraic thinking

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## 1 Introduction

Algebraic thinking is argued to be a key ability that children need to develop from an early age for their understanding of formal algebra in later years (Venenciano et al., 2020). In many countries, curricula and mathematical policy documents stipulates a teaching that promotes the youngest students’ algebraic thinking (Cai & Knuth, 2011, see also Venenciano et al., 2020). Kieran et al. (2016, p. 1) explains that “[m]athematical relations, patterns, and arithmetical structures lie at the heart of early algebraic activity”. At the beginning of 2000, four ways of addressing the issue of early algebra were defined as “(i) generalizing related to patterning activity, (ii) generalizing related to properties of operations and numerical structure, (iii) representing relationships among quantities, and (iv) introducing alphanumeric notation” (Kieran et al., 2016, p. 5). Representing relations among quantities as a teaching model refers to a curriculum developed by El’konin and Davydov (ED Curriculum) in which students’ understanding of part-whole relationships is at the core (Schmittau, 2003, 2004, 2005). The ED Curriculum, with its roots in the cultural





historical tradition of Vygotsky (1987), Leontiev (1978) and Galperin (1968), has been described within the research field of early algebra as a curriculum model with potential for developing young students' algebraic thinking (e.g. Dougherty, 2004; Kaput, 2008; Kieran et al., 2016; Carraher & Schliemann, 2014; Venenciano et al., 2020). However, Kieran et al. (2016) argue that more research and empirical examples of how such a curriculum can be realised in an everyday teaching setting are necessary. This article seeks to contribute with such an example based upon a case study depicting how some aspects of equations can be theoretically explored and reflected upon by young students. The students were invited to use a graphic model as a mediating tool (a learning model) in a teaching situation framed by the curriculum designed by El'konin and Davydov and its complementary learning activity theory (Davydov, 2008; Repkin, 2003; Schmittau, 2003, 2004).

In the two following Sections (1 and 2) we provide the framework for our aim and research questions. In Section 3, we provide a more detailed description of the learning activity and its central concepts. Methodology is presented in Section 4, followed by the result, divided into two parts and presented in Sections 5 and 6. The article ends with concluding remarks in Section 7.

## 1.1 Early algebra – realising a written curriculum

The field of early algebra is interested in the development of students' algebraic thinking and problem-solving abilities (Kieran, 2018; Radford, 2012, 2018; Radford & Barwell, 2016; Warren et al., 2016). This is sometimes related to teaching in which students are to be engaged in algebraic or theoretical work (Kieran et al., 2016). In developing these skills and abilities early, some researchers believe they are tackling a known problem with the commonly-used arithmetical foundation of algebra (Kaput, 2008; Lins & Kaput, 2004; Radford, 2006, 2010). As previously mentioned, the ED Curriculum is regarded as a promising alternative route when attempting to alter a teaching tradition that introduces students to algebra based on an arithmetic approach (Carraher & Schliemann, 2014; Kaput, 2008; Kieran et al., 2016).

### 1.1.1 Teaching for algebraic thinking

For very young students, the ED Curriculum comprises a series of deliberately sequenced problems of measurements that require students to expand known problem-solving methods and tools to develop their understanding at a theoretical level (Davydov, 1962, 2008; Schmittau, 2004, 2005; Sophian, 2007; Zuckerman,

2004, 2005). The idea of introducing numbers and mathematical operations through measurements is thus central. Schmittau (2004) argues, in line with Vygotsky, that relational analysis of quantities must precede the development of the concept of numbers. In a discussion of algebraic thinking, Schmittau and Morris (2004) claim that the ED Curriculum:

[d]evelop[s] children’s ability to think in a variety of ways that foster algebraic performance. First, it develops theoretical thinking, which according to Vygotsky comprises the essence of algebra. For example, the children develop a habit of searching out relationships among quantities across contextualized situations, and learn to solve an equation by attending to its underlying structure. ... Their ability to interpret a letter as “any number” allows the teacher to introduce children to the kind of general argument that is the hallmark of algebraic justification and proof. (Schmittau & Morris 2004, p. 23)

Kieran (2004) provides the following definition of algebraic thinking:

Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. (Kieran, 2004, p. 149)

Radford (2018, p. 8) highlights that a definition of algebraic thinking such Kieran’s also needs to include a requirement that the students be able to treat “indeterminate quantities in an analytical manner.” Thus, teaching aimed at developing algebraic thinking must support an analytical approach. Radford (2012, p. 119) argues that

[w]ithin the theory of knowledge objectification, thinking is considered a relationship between the thinking subject and the cultural forms of thought in which the subject finds itself immersed. More precisely, thinking is a unity of a sensing subject and a historically and culturally constituted conceptual realm where things appear already bestowed with meaning and objectivity.

### 1.1.2 Teaching and development of theoretical thinking

The ED Curriculum draws theoretically on Vygotsky’s idea that “teaching should take a leading role in relation to mental development” (Chaiklin, 2003, p. 169). From this perspective, the development of theoretical thinking requires a specially-organised practical activity – a learning activity in which students can reconstruct mathematical concepts, norms and values and thus learn to master culturally and historically developed theoretical ways of knowing. In mathematics, theoretical thinking is often



exemplified by algebraic thinking (Krutetskii, 1976; Radford, 2021). Central to learning activity is the idea of ascending from the abstract to the concrete (Davydov, 2008). He claims that, if students first work theoretically on an object of knowledge to find embedded structural and general aspects of a concept as well as its conceptual relations, they can later find concrete instances of the theoretical knowledge. Dreyfus (2015, p. 117) argues:

According to Davydov’s ‘method of ascent to the concrete,’ abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form.

However, the abstract structural and relational aspect of an object of knowledge is not available to the students through a teacher’s direct instruction (Davydov, 2008; Schmittau, 2004), and thus, in realising a learning activity that enhances students algebraic thinking, a mediating tool – a learning model<sup>1</sup> (Gorbov & Chudinova, 2000) – that students can manipulate, change and examine when elaborating on and discerning the abstract content of an object of knowledge is necessary. Within learning activity theory, a learning model “fixates the universal relation of some holistic object, enabling its further analysis” (Davydov, 2008, p. 126).

## 2 Aim and research questions

Education realised through tool-mediated learning activities is thus a foundation of the ED Curriculum. However, realising this type of teaching places substantial demands on the teacher when, for example, designing tasks, initiating a problem situation or supporting student collective theoretical reflective work in the classroom (Kieran, et al., 2016). Even though there is research within the field of early algebra that seeks to develop teaching in line with the ED Curriculum (e.g., Dougherty, 2004; Schmittau, 2003; Sophian, 2007; see also H. Eriksson, 2021; I. Eriksson et al., 2021), we still do not have a substantial body of empirically-based knowledge about how to realise such teaching. Furthermore, there are few empirical examples of how teachers in a Western context can use learning models and collective reflections to support

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<sup>1</sup> A learning model must not be understood as a mathematical model but a form of tool for visualising and elaborating core ideas.

student learning of algebraic ideas. There are even fewer empirical examples of how the ED Curriculum is realised in the experimental school – School No. 91 where it was designed (see below).

Given this, and based on a case study from School No. 91, our aim is to provide a concrete example of how a specifically-designed teaching can promote the youngest students' algebraic thinking. The aim is also to analyse which algebraic or structural aspects of equations are made available when the students and their teacher collaboratively uses a learning model as a mediated tool in a learning activity. The analysis is guided by the research questions (RQs):

- RQ1: What algebraic thinking on the relationship of the whole and its parts and the unknown in equations, can be discerned through a learning model in a lesson framed by principles of learning activity?
- RQ2: What, in student and teacher tool-mediated joint action, promotes exploration of the algebraic aspects of equations?

### 3 Learning activity

Learning activity theory must be understood in relation to specific theoretical content. For example, the ED Curriculum, as it is known in the West, is designed to realise learning activities in mathematics (Davydov, 2008; Dougherty, 2004; Schmittau, 2003, 2004; Schmittau & Morris, 2004; Sophian, 2007; Venenciano & Dougherty, 2014; Venenciano et al., 2020). The basis of the curriculum for the youngest students is measurement and units of measurement. This curriculum was developed experimentally at School No 91 in Moscow where, in 1958, El'konin and Davydov, in line with Vygotsky theoretical assumptions (Davydov, 2008), began their experimental research on the influence learning processes exert on student cognitive development.<sup>2</sup> Based on their experiment, El'konin, Davydov and their team proposed new content and new methods for learning and teaching mathematics and language in primary school.

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<sup>2</sup> As aforementioned, a learning activity is theoretically built on Vygotsky's (1987) cultural historical theory and Leontiev's (1978) activity theory. Thus, Davydov and El'konin further developed the work begun by Gal'perin (1968) and formed two learning activity curriculums for reading and writing and mathematics, respectively.

### 3.1 Learning activity and learning models

In learning activity, the overarching goal is the development of student theoretical thinking and agency, that is, their capability to act and participate in a new and independent manner in different content-specific activities (Davydov, 2008; El'konin, 1999; Repkin, 2003; Rubtsov, 1991; Zuckerman, 2003). In order to invite the students into a learning activity, the teacher usually introduces a problem situation (Repkin, 2003) which must contain some abstract but central structural or theoretical aspects of specific content (an object of knowledge) that the students need to become conscious of. The teacher cannot merely present a problem and tell the students to solve it. In order to become involved in a learning activity, students must, through analysis of the situation, develop a motive for engaging in the activity, and then transform the problem into a learning task. The first step of student analytical work includes joint reflection on what previous knowledge and known tools (i.e. learning models) they can test (Davydov, 2008; Rubtsov, 2013; Zuckerman, 2004). Repkin (2003, p. 27) explains that students need “new modes of actions”. Students must transform the initial problem situation into a learning task that implicitly leads them to discover new methods, or new tools, to solve the problem and the teacher encourages them to collectively reflect upon and defend and expand their solutions. The discussion does not end until the students have reached a conclusion they consider correct or plausible (Davydov, 2008; Schmittau, 2003, 2005; Sophian, 2007). However, the youngest students must learn how to work within a learning activity and are thus dependent on the teacher as a more knowledgeable other (Vygotsky, 1934/1963, 1987). A learning activity can make it possible for students to work within what Vygotsky (1934/1963) described as a zone of proximal development (ZPD).

To make it possible for students to explore “the abstract” of a specific object of knowledge, Davydov and his fellow researchers suggest that each learning activity must be realised with the help of learning models as visual mediating artefacts. The purpose of a learning model is to visualise the structural aspects of an object of knowledge and make it possible for the students to manipulate it during their analytical work. A learning model can take the form of a scheme, for example, depicted by line segments (such as in this article), or as a semiotic system, as for example,  $A=B+C$ . Davydov (2008, p. 95) explains that

the structure of semiotic systems reproduces or copies the structure of the object. For example, a chemical formula has semiotic mediated function since

its connections and sequence of elements convey the character of an actual chemical relation, the structure of a chemical compound. Of course, as in any other form of model, this reproduction is approximate, simplifying and schematizing the actual object.

A learning model may also be in a physical form, but in that case is mostly in combination with a symbolic model on the blackboard (Gorbov & Chudinova, 2000).

#### 4 Data and methods of analysis

In this article, we present the results of a case study (Flyvbjerg, 2011; Yin, 2014) conducted in School No. 91 based on Inger Eriksson's (Author I) visits to the school. Each visit, in 2013, 2016, 2017 and 2019, consisted of 5–9 hours of classroom observations in mathematics, a total of 27 lessons. Most of the lessons observed were conducted in the primary grades and several were taught by Ms Natalia Tabachnikova (Author II). As is characteristic of a case study, there were multiple sources of data (Merriam, 1998). All classroom observations were documented using video recordings, complemented with digital photos of student work and the blackboard text, and audio recordings or field notes from formal and informal follow-up discussions with the teachers, especially those who taught mathematics. During each lesson observed an interpreter provided in situ Russia-to-English, while some members of the local research team also attended the lessons and complemented with contextual comments. The main interest of a case study is what can be learned from a specific case or more precisely, how to gain new insight into local practice (Flyvbjerg, 2011). In this case, the interest was the function of the learning models in the students' collective exploration of structural aspects of equations. By choosing a single lesson, it is possible to make a more detailed analysis of the tool used and constituent actions. In order to understand what learning is made possible in a particular situation, for example during a single lesson, it is vital to become familiar with the daily teaching in a broader sense (Stake, 2005). The use of a learning model and the communicative actions in the lesson chosen is considered as representative for the lessons observed in total.

On the one hand, Author II, who taught the lesson chosen as the example for this article (see below), has an insider perspective on the practice analysed in this article. On the other hand, Author I, through her recurring visits to the school, has gained an outsider perspective. During the visits, Author I had several opportunities to discuss the principles of teaching, and the learning activity theory together with Author II, her

colleagues and the local researchers. Further, the first author is familiar with the ED Curriculum through her own research (see e.g. Eriksson et al. 2021; Eriksson & Jansson, 2017; Eriksson et al., 2019; Wettergren et al., 2021). Milligan (2016) addresses the issue of researchers' positioning as an insider or an outsider and suggests that it is possible to develop a position of an inbetweener. We find this concept useful when describing our collaboration.

#### 4.1 Data

The data for this article is from the observation of a typical ED Curriculum first grade lesson (7–8 years old). The 45-minute lesson was video-recorded by Author I in April 2017, when the first graders had been in school for approximately 7–8 months. Of the 23 participating students, 12 were girls and 11 boys. As a complement, some of the student worksheets were photo-documented. The video recordings, as the main source of data, were translated and transcribed into English by a researcher familiar with School No. 91 and learning activity theory. Author II reviewed the film and the first draft of the transcript. Finally, the transcript was jointly reviewed, and clarifications made, by Author I and Author II. The transcription captured all oral communication, complemented by gestures and intonation in situations where they provided meaning (Radford, 2010; Roth & Radford, 2011). The transcription was verbatim, speech neutral and organised dialogically (Linell, 1994). In the translation, nuances of the classroom interactions may have been missed in some cases (see Radford, 2010; Roth & Radford, 2011). Unfortunately, the sound quality was not always optimal, which may also have led to omissions, and the classroom atmosphere was not easy to capture in a transcript – at times the students were unable to sit still and wait to be called on. The atmosphere was intense and lively. To compensate for this, we repeatedly reviewed the transcript, the translation and the video/photo documentation of student actions, gestures and facial expressions.

Central to this lesson was the learning model depicting an algebraic structure of a whole and parts in the form of a line segment scheme  $|—|———|$  with which the students were familiar. The 'whole' was marked with a curved line on the upper or lower part of the model, and the parts were correspondingly marked with two shorter curved lines (illustrated below). In this lesson, the line segment model was presented in three drawings on the blackboard. Author II explained, in line with the ED Curriculum, that the overarching aim of the lesson was to stimulate student analytical and theoretical thinking, in this case in relation to the algebraic structure of equations.

## 4.2 Analysis

The results of the analytical work are divided into two sections (5 and 6). In order to establish an empirical foundation for analysis in relation to the two research questions, a narrative of the unfolding learning activity was constructed which comprised five identified sequences that captured the key events during the lesson. To identify the beginning and end of a sequence, focus was directed towards the teacher's communicative verbal and non-verbal actions that signalled such transitions, e.g. saying: "Look at the blackboard, please" [while she puts one forearm on top of the other—a known signal that students should be quiet]." Of the five identified sequences, three (first, third, and fourth) were the focus of this analysis. The second sequence was omitted due to silence while students worked individually. The fifth sequence was omitted mostly due to space constraints but also since this sequence repeated much of the action in the already selected sequences. Sequence 1 (approximately ten minutes) involved an introduction to the problem situation with the discussion prompt, *Three drawings: What is similar?* In Sequence 3 (approximately six minutes) the students wrote an equation for their problem: *Writing a programme for a calculator*. Finally, in Sequence 4 (approximately six minutes), the students reflected on the puzzling fact that there were three equations on the board but several problems presented by the students: *Three solutions but several problems*. The lesson concluded with Sequence 5, an additional task in which the teacher wrote  $120 - x = 15$  on the board and asked the students to visualise this equation using the same line segment learning model from the previous task. An engaging discussion based on this new equation followed but is not included in this article. The narrative of the three chosen sequences is presented in Section 5. The second step in the analysis, presented in Section 6, aims to provide a more elaborated answer to the two research questions.

The analysis of the empirically based narrative was inspired by concepts related to learning activity (Davydov, 2008). From that perspective, human actions are understandable if it is possible to discern who does what (what are they doing), why (the goals of the actions), and with what tools (implied that all actions are tool-mediated)? In such an analysis, both oral and written speech, combined with the teacher and student intonations and gestures, provided analytical information when trying to capture what constitutive tool-mediated, goal-directed actions are occurring (Roth & Radford, 2011). In relation to research question 1, special attention was paid to which understanding of the constituent parts of an equation was made available



through the joint tool-mediated actions. In relation to research question 2, special attention was placed on what in the tool-mediated joint actions enabled discernment of structural aspects.

### 4.3 Ethical considerations

Because School No. 91 is an experimental school, parents of students enrolled there provide consent for researchers and teachers to experiment with the curriculum, observe and videotape lessons and to study student learning. The researchers and teachers were permitted to use the videotapes and results of this study for only two purposes: scientific articles and teacher-training courses. In the transcripts, pseudonyms were used for the children and photographs were selected or retouched to reduce the possibility of identifying individual students. Individuals who know the students may, however, recognise them.

## 5 A narrative of the unfolding learning activity

The following sequences from the chosen lesson were described narratively and chronologically as the activity unfolded.

### 5.1 Sequence 1. The problem situation built into the three drawings: what is similar?

As the students enter the classroom, there are three drawings on the board (Figure 1), each based on the type of line segment model that the students are familiar with.

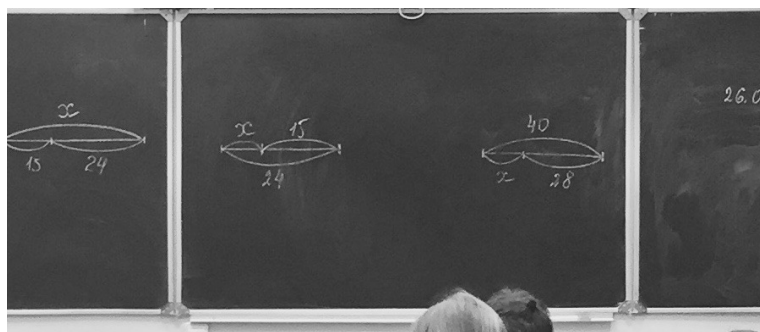


Figure 1. The three drawings on the board

The teacher asks the students to compare the three drawings and try to determine what they have in common. There is eager mumbling and several students raise their

hands. The teacher asks a student to come forward and indicate her suggestions. The first thing that the student identifies is the  $x$  in each drawing and the teacher checks that everyone agrees on this assertion.

Excerpt 1.

- Teacher: Please look at the board. We see three drawings. What do you think these three drawings have in common? ...
- Katya: [Goes up to the blackboards and points] There is  $x$ .
- Teacher: Who agrees with Katya? Aha... All saw it. Wonderful! What else is common in all three drawings?
- Students:  $x$
- Olechka: The three drawings have an  $x$
- Teacher: Olechka, what is  $x$ ?
- Students: Unknown.
- Teacher: The unknown. Dimitra, have you noticed anything else? Mila, what have you found?

The teacher asks for other similarities and various students come forward to show what similarities they have found, some mention the numbers in the drawings, others the structural aspects of a whole and two parts. The teacher then signals verbally and with gestures that there can or must be more similarities.

Excerpt 2.

- Stepka: Look, they are all similar! Here, they have a large part, a medium part and a small part [shows the first drawing]. Here is a large, medium and small [second drawing] and here too [third drawing].
- Teacher: Good. And you, what do you want to show us? ... Mila? You also found something they have in common ...
- Dimitra: I realised that in all three drawings the whole consists of two parts. In this they are similar.
- Teacher: So, children [turning to the class] do you understand what Dimitra means?
- Dimitra: I wanted to say that there is a whole and two parts in all the drawings.
- Teacher: Do you agree with that?
- Children: Yes, yes... but Stepka did say that...
- Teacher: I think Stepka said something else? Right Stepka?
- Stepka: I said all had a large, a medium and a small.
- Teacher: Yes. And Dimitra said that there is the whole, which consists of two parts. And all three are like that.

The teacher is obviously satisfied when Dimitra identifies the algebraic structure of the relationships between the whole and the parts in the different drawings. A structure can be expressed in various ways, as for example,  $a+b=c$  or schematically as

in the drawings. Using what can be seen as an imaginary playful format she then asks students to examine the structure in more detail: “You know, one boy from another class said one thing... He said that he thinks that two drawings are much more similar to each other than the third one. Which one is different?” During this sequence, several students come to the board simultaneously – all engage in explaining and arguing. Some students work together with the teacher, and some work in pairs (Figure 2).



Figure 2. The teacher signals “I don’t understand.” Some of the students give an explanation and the teacher signals that she does not understand by lifting her shoulder and holding out her hands in a questioning gesture as she says: “Can that be?” or “Is this right?”

### Excerpt 3.

Teacher: Raise your hands, please, who found the two drawings and sees their similarity and how the third is different? Michail. [Michail goes to the board and points to the first and second as similar]. So, these two [teacher points] are similar, and this one is completely different? [Teacher turns to the class and asks...] Yes? Who understood what Michail means? [the intonation in the teacher’s voice suggests that she doesn’t understand] ... Who can show and explain what these two drawings have in common? [A boy goes to the board and indicates ...]

A student: Here is [pointing]  $x$ , and here is  $x$ , here is 24, and here is 24.

Teacher: Michail. Did you mean this? [Michail nods]

Teacher: And who was thinking of something else?

On several occasions, the teacher involves the whole class by saying, for example, “Did he guess correctly?” and turns away from the students who have given the suggestion. The students are apparently used to participating in such collective discussions characterised by signals and gestures related to “agree” or “don’t agree”.

## Excerpt 4.

- Varya: [Pointing to the first and second drawing] If we turn them over, it will be the same...
- Teacher: Have you also thought about these two? [pointing to the first and second drawing].
- Varya: Yes.
- Teacher: Varya agrees with you. And who had the other two [drawings] in mind? ... What do you think? And you? And you? [the teacher addresses different children. A girl goes to the board and indicates the other two drawings – the second and the third].
- A student: These [pointing to the different parts – each line segment – of the second and third drawing] are similar, because here  $x$  is large [first drawing] and here [second and third]  $x$  is small.

From the video it is possible to note that many (if not all) students are involved: some are standing, while others have their hands raised and are eagerly calling out their suggestions. The atmosphere is intense and engaging.

## Excerpt 5.

- Student: [Several students are at the board, pointing and explaining] These are similar because here  $x$  is large [first drawing] and here [second and third]  $x$  is small.  
[Another girl goes to the board and they both indicate]: Here  $x$  is big, and here, and here it is small.  
[A third girl comes up to them and says, while pointing to the second and third drawing...]
- Olechka: Here the  $x$  is a part [second and third drawing], and here the  $x$  is the whole [first drawing]
- Katya: [Pointing to some numbers] And here and here it is four.
- Teacher: Yes. Four is good. But it's more important to understand where  $x$  is the whole, and where it is the part.

The episode ends, and the teacher gives the students an assignment related to continuing their exploratory work with the learning models. The students are asked to copy one of the drawings on the board into their workbooks, but without telling anyone which one. When they have copied one of the drawings, the teacher asks everyone to write a story (see below) in relation to their chosen drawing – a problem in the form of a story using the whole and its parts in their drawing. When observing the students as they started to write their stories it was obvious that they were used to this type of work.

## 5.2 Sequence 3: A programme for a calculator – finding a concrete solution

After a while, the teacher invites several students to the front of the class to read their stories. In relation to the second drawing (see [Figure 1](#)), one of the students read:

On his birthday Peter was presented with 15 new cars [Hot Wheels]. And now he has 24 Hot Wheels cars. How many Hot Wheels cars did Peter have before his birthday? (“Story” read by a student)

In relation to the third drawing (see [Figure 1](#)), another student read:

There were 40 children on the school bus. 28 children are seven years old, the rest are eight. How many 8-year-old children are there on the school bus? (Story read by another student)

Next, the teacher asks the other students to guess which drawing the story is about. The duration of this process is approximately eight minutes, after which the teacher again calls for the students’ attention. Under each drawing, she draws “ $x =$ ” and asks the students to do the same under their chosen drawing.

Excerpt 6.

Teacher: How will you find  $x$ ? And, we need to make an action plan i.e., write an equation such as  $x = a + b$  or  $x = d + e$  or  $x = 8 - 5$ . If someone cannot calculate the result, that’s not a big deal. We will choose a student who counts well, someone who will be a ‘calculator’ and they will count for everyone who has difficulties. For us it is important to just make an action programme for a calculator, all right?

With these instructions, student work takes a new direction. The teacher wants the students to write an equation for each of the three drawings that could be used to program an [imaginary] calculator, stressing that it is not necessary to figure out the answer, simply to write the ‘program’ in relation to the drawing they have chosen and the problem (story) they have written. In doing so, she uses what van Oers (2009) describes as a playful format to manage the fact that not all the students are able to solve the equation. The students immediately seem to grasp the imaginary calculator idea.

## Excerpt 7.

- A student [calls out]:  $24 + 15 = 39$  [pointing to the middle drawing on the board].
- Teacher: You have already calculated!  $24 + 15 = 39$ . Stand up, please, those who also have [points to the middle drawing]. There are so many of you! Two... no, four. Varya, please, what have you written?
- Varya:  $x$  is 15... [stops].
- Teacher: So, you think an unknown [ $x$ ] plus 15 is 24?
- Olechka: No! [protesting] May I? [goes to board].
- Teacher: OK. Write.
- Olechka: [Writes and says]  $24 - 15 = 9$ .
- Teacher: Aha. Look here, what Varya meant: how much should I add to 15 to get 24? But you did not write a programme for the calculator. How much should I add so that I get...? We should write a programme for the calculator to make it clear.
- Mila: [Now two girls are writing—almost on top of each other—on the board]  $x = 24 - 15 = 9$  ...

The teacher then asks for the students who have written an equation and a solution to the second and third drawing.

## Excerpt 8.

- Teacher: Ok. Thank you. Now, who was solving this one [referring to the second drawing]? ... Michail, come to the blackboard. Dina, you've been here already... ok, you may come and support Michail. [The boy writes behind the " $x =$ " that was already on the board]:  $x = 40 - 28$ .
- Teacher: Let Mila continue.
- Mila: 18 [writes " $=18$ "].
- Michail: [Turns from the board to the teacher and says quietly while signalling with his finger] "I don't agree". I think it is 12.
- Teacher: You think it is 12... [looks inquiringly at the class]? [Children nod affirmatively and signal consent].  
[Mila writes 12]
- Teacher: [Turning to Mila] Don't worry. You wrote the correct programme for the calculator. That is very important. And it will help us calculate. Thank you!
- Teacher: You have written very different programmes for the calculator with very different numbers and different answers. Here, two groups wrote minus in the programme for the calculator [points to two drawings on the right], others wrote plus [points to the drawing on the left]. What do you think? Why? Explain to me, please, when to subtract, and when to add.

At the end of the exercise, the teacher asks the students how it is possible that some of the programmes they have written for the calculator use minus and some use



plus, which gets them to consider the algebraic relational structure of the whole and its parts and possible operational functions.

### 5.3 Sequence 4: Three solutions but several problems

The students are fully occupied with their assignment of writing programs for the calculator and explaining why subtraction or addition is required in some programmes, when the teacher gives them a question about how many problems they have created together and how many solutions are possible. At first, the students apparently struggle to understand what the teacher is asking for.

Excerpt 9.

- Teacher: ... What do you think? How many solutions are written on the blackboard? [referring to the three equations with their respective solutions that the students have calculated on the board]. Show me with your fingers, how many. [Several students show three fingers in the air; the rest join them]. I can even count them. One, two, three... Yeah. And how many problems have we designed altogether [referring to the stories that the students had created earlier]? ...
- Teacher: If you want to answer, raise your hand. Why did this happen? Were there 12 (or 15) tasks and only three solutions? 12 tasks, then 12 solutions?
- Students: [In chorus] No!
- Teacher: Well ten at least...
- A student: [Approaching the board with hesitation] Because every task has one solution!
- Teacher: I don't understand. One solution? But here are three of them. Look: one, two, three. But there were 15 problems! How did this happen?
- A student: Because there are many people in the classroom!
- Teacher: There are many people in the classroom, that's why there are 15 problems, but solutions?
- A student: Everyone has their own answer!
- Teacher: Aha, so we have 15 answers?
- Students: No...
- Teacher: No... That is what I'm talking about. So, we have 15 problems and only three answers. Why?
- Student: Because everyone has his own answer, everyone wants to share his own knowledge!
- Gavril: Because we composed our problems for these three drawings. Therefore, there are three solutions...
- Teacher: I really liked what Gavril said. So, someone came up with a puzzle [a story] for this drawing [pointing to the first drawing], other children for the second [pointing to the middle drawing]. And which of you wrote a story for this drawing [points to the drawing on the right]? Therefore, we got only three solutions for many

different problems. [Finally, it seems the children realise how they got only three solutions for the 15 problems they compiled].  
 Teacher: Well done. Everyone did a good job.

In this episode, the teacher puts forward a problem that is difficult for the students to understand. “How can there be so many problems but only three solutions?” The teacher repeatedly encourages the students to provide an explanation for the *mysterious fact* that there are several problems or stories but only three solutions. First, when the teacher asks which students have created a story for the first drawing, and which for the second and third, it is possible for them (or most of them) to understand that one equation with its solution can match several concrete problems or stories.

## 6 Utilising young student algebraic thinking about equations

In this section, we analyse the narratively-depicted learning activity and its evolving in relation to the two research questions.

### 6.1 Algebraic thinking of the relationship of the whole and its parts and the unknown in equations

The first research question addresses the idea that algebraic thinking of the relationship of the whole and its parts and the unknown in equations can be discerned through a learning model in a lesson framed by learning activity.

In the basic line segment learning model exemplified in the three drawings (see [Figure 1](#) above) the selected numbers and the placement of the  $x$  was important for making algebraic ideas related to equations possible to collectively discern and reflect upon. In one of the drawings,  $x$  represented the whole, and in the other two drawings it represented a part of the whole. How the teacher posed the questions and how she let the young students contribute different suggestions supported by gestures and language played a critical role when the students explored the three drawings. This made it possible for the them to discern that:

- a) The symbol  $x$  can be used to symbolise something unknown that can be either a whole or one of the parts.
- b) The problem embedded in the three drawings mathematically describes a relation between the whole and its parts (i.e. what the learning model with its line segments and the arches depicts).

In the following sequence, the students first secretly chose one of the drawings, wrote a story in relation to it, and read it aloud so that the other students could guess which model the story was written to describe. The students then wrote a “program for a calculator,” that is, they wrote an equation that modelled the relationships. The teacher emphasised that it was not necessary to find the answer to ‘the program’ because the imaginary calculator could do that. However, we observed that all the students managed to find the answer to their equation. Using the learning model in this manner, the students had the opportunity to discern at least the following that:

- c) A story or a visual representation (i.e. the learning model) can be ‘translated’ into an algebraic equation.
- d) In an equation, the relational structure between a whole and its parts may vary, and the unknown can be either the whole or any of the parts (the exemplification of the learning model in the three drawings made it possible for the students to discern this).
- e) A problem, when translated into a mathematical problem as a first step towards a solution can be formulated as an equation that will make it possible to determine the unknown – the value of  $x$ .

In the third sequence, when the students had written the programme for the calculator in their workbooks and on the board and calculated  $x$  in the three drawings, the teacher confronted them with a new problem. She asked the students how there could be so many problems (in the different stories created by the students) but only three solutions. In this situation the teacher addressed a topic that was apparently difficult for the students to figure out. The teacher, however, was persistent and posed the questions several times in various ways even though it seemed as if the students had more or less provided the same type of explanation. First, when the teacher called students to the board, and then in relation to each of the calculations the students demonstrated that the question could have an answer, some of the students wrote their problem for only one of the drawings. This contradictory question from the teacher made it possible for the students to discern that:

- f) An expression or equation can represent different contextual problems or situations.

This can be described as an emerging algebraic thinking of the generality of equations and thus a first step in being able to ascend from the abstract to the concrete.

## 6.2 The student and teacher tool-mediated joint actions

The second research question addresses what in the student and teacher tool-mediated joint actions promotes exploration of the algebraic aspects of equations.

The three original drawings on the board with the question “How are all these three drawings similar?” can be considered the introduction to the first problem situation. The students were to identify and analyse different relational aspects built into the problem situation with the help of the learning model used in the three drawings. The analytical or theoretical work was conducted jointly, and the students apparently challenged each other to find more similarities.

- g) Student theoretical work was collectively realised by those at the board and the others who remained at their desks through (previously agreed-on) hand signals of agreement or disagreement. Several students also verbally expressed whether they agreed or not. This joint labour, as Radford (2018) describes it, made it possible for the students to both see and hear others’ suggestions and explanations while simultaneously expressing their own understanding. This can be described as a collective reflection (Zuckerman, 2004).
- h) While students at the board used the learning model and its components to make their thinking and suggestions accessible to others assessment the teacher often acted *as if* she did not really understand what the students were trying to say and mostly signalled this with gestures and by asking other students to explain. This promoted the students to elaborate the content further.
- i) The teacher’s ‘unwillingness’ to understand, combined with the way the three drawings were designed (based on the line segment learning model), allowed the students to elaborate on the algebraic ideas of the whole and its parts, and  $x$  symbolising the unknown. Understanding that the unknown symbolised by an  $x$  can be any part of an equation, the whole or one of the parts.

To summarise, this can be described in terms of materialising student collective algebraic thinking (Radford, 2006, Venenciano et al., 2020). The learning models and the problem situation, combined with the communication prompted by the teacher, made it possible for the students to reflectively take the others’ position while simultaneously better understanding their own ideas (Zuckerman, 2003, 2004).

## 7 Concluding remarks – teaching that enables and enhances algebraic thinking

The learning activity realised in second author's classroom can be described as analytical and reflective. The structure and use of the learning model combined with the teacher's prompts and her ability to take advantage of student answers and questions, created opportunities for the students to analytically reflect upon others' suggestions and explanations. That is, the students could use each other's thinking (visualised with the help of the learning model) to further their own thinking (Zuckerman, 2003, 2004). Furthermore, the opportunities for the students to act and express their ideas and to have these elaborated by others appeared to promote the development of their agency (Davydov et al., 2003).

Following students' joint actions from Sequence 1 through Sequence 3, there are indications that they increasingly expressed themselves analytically and mathematically (Radford, 2018). First, the three relational aspects embedded in the drawings were not discerned by the students. They talked about smaller and bigger parts, but not of how they were related to each other. Thus, the students did not initially reflect upon what the whole was and what the parts were and what in that structure was known and what was not. Second, the analytical work that was required of the students when asked to create a word problem and an equation for one of the drawings made a mathematically-relevant understanding possible. Given these aspects, because of the student participation in the learning activity, it was possible for them to develop complex relational thinking regarding, for example, possible structures of equations, the unknown and the relationship between equations and contextual situations. Regarding quantities – as mentioned in the introduction, Radford (2018) addresses the need to consider student analytical work as an indicator of algebraic thinking. If a student merely guesses or uses a trial-and-error strategy and produces relevant answers, this does not count as algebraic thinking. Thus, it seems plausible that the students had opportunity to develop complex relational thinking because of their participation in the learning activity.

In a learning activity such as that that evolved during this lesson, several aspects must occur simultaneously. Because the object of knowledge embedded in the problem situation and the learning model may at any moment, be at risk the teacher and student co-actions are significantly important. In particular, the teacher needs to consciously address individual student suggestions and explanations, making them available for the other students to continued exploration.

This study may be considered as limited in that only one lesson was analysed. However, the lesson chosen out of 27 observed lessons is representative in relation to the aim of teaching and the use of learning model as a mediating tool for students' problem-solving theoretical work (Larsson, 2009). Thus, we hope that our analysis can provide some indication that it is possible to illuminate algebraic ideas through the collective and reflective use of learning models. This may be regarded as a way to allow for complex relational thinking (Davydov, 2008) to take a materialised form that others are able to reflect upon (Radford, 2018, 2021; see also H. Eriksson & I. Eriksson, 2020; Eriksson et al., 2019).

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# Understanding “proportion” and mathematical identity: A study of Japanese elementary school teachers

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Studies have found that problems exist with respect to elementary school teachers’ understanding of proportions and their knowledge of the appropriate methods for teaching the concept. This study aims to help aspiring elementary school teachers form a healthy mathematical identity and deepen their understanding of mathematics. This quantitative study employed the descriptive-research survey method, surveying 86 students in 2019 and 110 students in 2021. Data were gathered using a survey questionnaire designed by Kumakura et al. (2019), with minor modifications made by the author. A major finding was that many students want to become elementary school teachers but are uncomfortable with the concept of proportions. Another important finding is that it is a challenge for students who wish to become elementary school teachers at a traditional school (University A) to hone their ability to use mathematical expressions and develop their sense of quantity. The findings suggest that it is important to help such students understand the content and refine their expressions.

## ARTICLE DETAILS

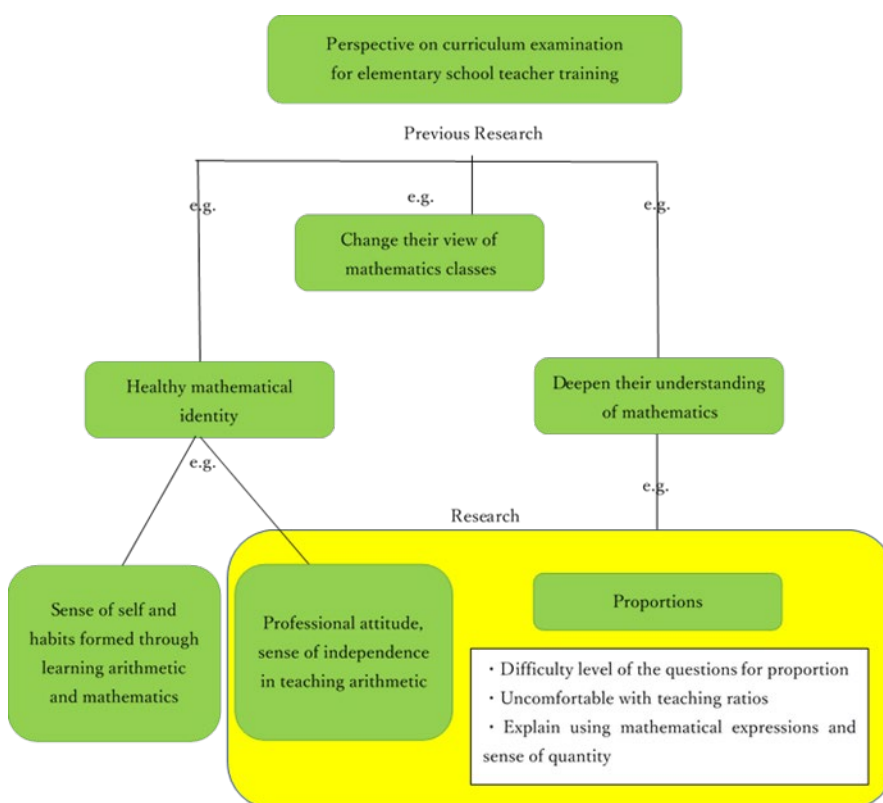
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## 1 Introduction

On December 26, 2019, an editorial in the *Mainichi Shimbun*, a daily newspaper in Japan, asserted that the sinking teacher applicant ratio is putting the future of Japanese children at risk: “*There were on average 2.8 applicants competing for each available position at Japan’s public elementary schools in this academic year’s employment exam cycle, tying a record low ratio set in 1991, according to the Education Ministry.*”

This demonstrates the high expectations regarding quality assurance in elementary school teacher training in Japan. In particular, the role of private universities is significant because, as of 2017, the number of universities with accredited programs for a first-class license to teach in elementary school stood at 183 private universities, 52 national universities, and four public universities. However, the teacher training programs at private universities in Japan face several challenges; in particular, few students choose mathematics as part of their university entrance exams. There is also a shortage of subjects related to mathematics and arithmetic at these universities, and the ratio between employment examinations and other examinations available to prospective teachers is decreasing. For these reasons, a method for examining the mathematics education curricula which is being used to train elementary school teachers at private universities in Japan is urgently needed.

In Japanese elementary school curricula, simple unit fractions, such as  $1/2$  and  $1/3$ , are taught in the second grade. The meaning of fractions and decimals—other than unit fractions—is taught in the third grade. In the fourth grade, addition and subtraction of equal denominators, doubling with decimals, addition and subtraction of decimals, and proportions for simple cases are taught. Multiplication and division of decimals, addition and subtraction of different denominators, proportions of two different quantities, and percentages are taught in the fifth grade. Finally, multiplication and division of fractions, proportions, and ratios are taught in the sixth grade. More specifically, in the fifth grade, students systematically learn multiplication and division of decimals, addition and subtraction of different denominators, the ratio of two different quantities, and percentages, whereas, in the sixth grade, students learn multiplication and division of fractions, proportions, and ratios. In the fifth grade, the concept of multiplication can be used in a broader range of situations and meanings by considering its relationship with division and when the multiplier is a decimal. In other words, students learn that  $A = B \times p$  (second usage). However, middle school students and beyond do not have any units specifically on proportions. Instead, they



work only on problems involving fractions, decimals, ratios, and proportions in units that focus on equations, shapes, and the use of data. Because these older students no longer study proportions, the level of understanding proportions among prospective teachers in Japan may be inadequate, which can affect their mathematical identity. This study seeks to explore and clarify the level of conceptual understanding of “proportions” among students who hope to become elementary school teachers in the future and who are attending a private university to undergo training and obtain suggestions for forming students’ mathematical identities.

## 2 Theoretical framework

### 2.1 Mathematical identity

Studies on identity in mathematics education include those that view identity from the perspective of participation and positionality—constructed through participation and involvement in social groups (Rabe & Wenger, 1993). Others regard identity as a narrative (Sfard & Prusak, 2005), while a final group includes affective constructs such as emotions, attitudes, and beliefs (Bishop, 2012). Aguirre et al. (2013) define mathematical identity as “*dispositions and beliefs about the development of the ability to use mathematics in mathematical and life contexts*” (p. 14).

In Japan, Takahashi (2020) defines this identity as the self-awareness of arithmetic and mathematics held by elementary school students, while Nishi (2017) hypothesized that a positive identity would be an outcome of mathematics education, in a study conducted among first-year university students in the faculty of education at Hiroshima University. This study examines mathematical identity among current and prospective elementary mathematics teachers in Japan in two contexts: the context of understanding arithmetic and mathematics for students who aspire to become elementary school teachers; and the context of their transition from being a student to a teacher. Mathematical identity is a concept that includes self-awareness of arithmetic and mathematics, the subjective awareness and sense of what the job of a teacher of arithmetic and mathematics entails, how one is performing or wants to perform, confirmation of identity through one’s occupation as a teacher, and professional attitude and ability to utilize and nurture one’s identity. Kambara (2021) defines the mathematical identity of students who want to become elementary school teachers as “a sense of self and habits formed through learning arithmetic and mathematics, as well as a professional attitude that makes the most of one’s own personality and sense of



independence in teaching arithmetic as an elementary school teacher” (p. 334). It is also necessary to understand mathematical identity as a concept that includes confirmation of identity through one’s profession as a teacher and professional goal, to maximally benefit from and nurture the identity. This study identifies the following needs: (1) to help students develop a healthy mathematical identity that allows them to positively experience mathematics and find new ways to learn the subject; (2) to transform their attitude about mathematics from a teacher-driven, standardized view of the classroom; and (3) to deepen their understanding of mathematics.

In this study, I use the case of “proportions” to explore some of the issues pertaining to points (1) and (3) in more depth. First, the teaching of proportions has been a longstanding issue in arithmetic education in Japan, and despite the availability of many studies and practices, a few problems with respect to the understanding of proportions have been noted (Kumakura et al., 2019; Yoshizawa, 2019). Second, for those who aim to become elementary school teachers, understanding the meaning of proportions and its appropriate teaching methods is mandatory. The purpose of this study is to clarify and investigate the level of understanding regarding the concept of “proportions” among students who want to become elementary school teachers and are undergoing training at a private university and obtain suggestions for the formation of students’ mathematical identities.

## 2.2 The concept of proportions

In mathematics, the concept of proportions is fundamental to many topics. A proportion,  $p$ , expresses the number of times when quantity  $A$  is compared with quantity  $B$ , where  $A$  and  $B$  are two similar types of quantities.  $B$  is called the base quantity, and  $A$  is the quantity to be compared. Proportions include the following relations:

- $p = A/B$  (first usage),
- $A = B \times p$  (second usage),
- $B = A/p$  (third usage).

The topic of proportions includes sub-topics, like percentages and “buai,” and number representations, such as decimals and fractions. To express  $A$  as a percentage of  $B$ , base quantity  $B$  is considered in terms of 100 units. A percentage is a ratio that compares a number to 100, and its symbol is %. Under the sub-topic “buai,” base quantity  $B$  is considered in terms of 10 units; and special terms like “wari,” “bu,” and “rin,” are used. In Japan, students learn that 1% is 0.01 and do not relate it to fractions. This

study will mainly focus on proportions expressed as percentages (%).

Understanding the knowledge level of mathematics teachers is an international endeavor. Even in the last decade, there has been much research on teachers' knowledge of proportions and ratios (e.g., Howe, 2013; Monteiro, 2003; Olanoff et al., 2014). In Japan, most studies on the semantic understanding of proportions have been conducted in elementary schools. A keyword search on CiNii for “understanding of proportions, junior high school students, high school students, and university students” demonstrated only two papers by Higuchi (2005) and Kumakura et al. (2019). The former was a study of college students, but it compared the results of one written test to find the rate of increase using the first usage of the ratio with the results of additive calculus of different denominators and fractions, and the basic knowledge of information literacy. It did not investigate the semantic understanding of percentages among college students. The latter study was conducted on junior high school and high school students, and the items were developed based on previous studies to investigate students' “deep understanding of proportion,” and detailed discussions were carried out. Following this study, I decided to investigate the situation regarding the understanding of the concept of “proportion” among students who want to become elementary school teachers through survey questions taken from the survey conducted by Kumakura et al. (2019; Tables 1 and 2).

### 3 Research method

#### 3.1 Measures

I investigated the situation regarding understanding percentages among students who want to become elementary school teachers. In doing so, I referred to the questionnaire and survey questions by Kumakura et al. (2019; Table 1). This quantitative study employs a descriptive-research survey method. The survey questionnaire by Kumakura et al. (2019) was used to collect data pertaining to students' understanding of proportions.

The questionnaire by Kumakura et al. (2019; Table 1) consisted of eight items (I) that measured understanding of “proportions” on the following dimensions: the need to understand proportions or utility of proportions (I1 and I2), the meaning of proportions (I3–I5), and attitude toward proportion problems (I6–I8). The investigator added a question on “confidence in teaching proportions,” so the final questionnaire had a total of nine items, which were used to investigate the students' mathematical

identities with a special focus on proportions. The author then compared the data from the college students with the data from high school students in Kumakura et al. (2019) to clarify university students' understanding.

**Table 1.** Mathematical identities with a special focus on proportions

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For the following nine items related to proportion, please select the option that best expresses your response to the questions:

(a) Strongly agree (b) Agree (c) Don't think so (d) Don't think so at all

- (1) The concept of percentages (%) is applicable to subjects other than mathematics.
  - (2) Knowledge of percentages (%) is necessary for daily life.
  - (3) If you express something as a percentage (%), you can see how much of the whole it is.
  - (4) Expressing results in terms of percentage (%) shows changes such as increases and decreases.
  - (5) We can compare two quantities by expressing them as percentages.
  - (6) Solving percentage (%) problems is fun.
  - (7) I am good at solving percentage (%) problems.
  - (8) I may try to solve problems in daily life by using the concept of percentages I have learned (%).
  - (9) I am confident in teaching percentages (%).
- 

Source: Author's addition to Kumakura et al. (2019)

The survey also included a questionnaire consisting of six major questions related to different types of proportions: Questions 1 and 2, respectively, correspond to the second and third usages described in section 2.2. Question 3 asks the respondent to contrast quantities. Questions 4 and 5 are pp-type questions (these are questions in which the reference quantity  $[A]$  is multiplied by the percentage  $[p]$  to obtain the comparison quantity  $[B]$ ; then, the reference quantity  $[B]$  is multiplied by the percentage  $[p']$  to obtain a new comparison quantity  $[C]$ ). Finally, question 6 is a p/p-type question (in this type, the comparison quantity  $[B]$  is divided by the percentage  $[p]$  to obtain the reference quantity  $[A]$ ; then, the comparison quantity  $[A]$  is divided by the percentage  $[p']$  to obtain the reference quantity  $[C]$ ). The questions are illustrated in Table 2.

Two survey rounds were conducted. In the first survey (2019), I analyzed the percentage of correct answers to explore the understanding of percentages among students who want to become elementary school teachers. The second survey (2021) used the same questionnaire as the first, and I qualitatively analyzed the writing of survey question 6. Through these surveys, I tried to get a deeper understanding of the

students' percentage understanding and gain a perspective to guide them. Furthermore, I tried to obtain suggestions for fostering their academic identity.

**Table 2.** Survey questions

Q1. If a cake with a regular price of 2000 yen is sold at a 30% discount, what is the price after the discount? Solve the question and mention all the steps.

Q2. Answer the following question:

A company is selling 180 g of canned salmon, the weight of the salmon is 20% more than that sold in the previous year. What was the weight of the canned salmon sold in the previous year?

(i) Solve the question mentioning all the steps, with special reference to how you calculated the content of the can sold in the previous year.

(ii) If you were to give an easy-to-understand explanation to a friend who did not understand how to solve the question, how would you explain it using a diagram, table, or figure? Draw/write this below. However, it is not necessary to use all the figures, tables, and pictures.

Q3. The following table shows the approximate land areas of Finland and Japan:

Country name	Finland	Japan
Land area	34 (million km <sup>2</sup> )	38 (million km <sup>2</sup> )

(i) Write an equation to find the approximate percentage of the land area of Finland with respect to the land area of Japan. However, you do not need to find the answer.

(ii) Write an equation to find the approximate percentage of the land area of Japan with respect to the land area of Finland. However, it is not necessary to find the answer.

Q4. At Junior High School A, 30% of the students commute to school by bicycle, and 60% of them are boys. What percentage of the school students are boys who ride bicycles to school? Solve the question with complete steps.

Q5. We looked at the annual number of visitors to the zoo from 2015 to 2017; the number of visitors in 2016 increased by 10% compared to the number of visitors in 2015. How did the number of visitors in 2017 compare to the number of visitors in 2015? Choose one correct answer from the options given below, circle it, and write the reason for your choice.

a. Increased b. No change c. Decreased

Q6. What is the ratio of forest area to total area in town A? Write the answer and how to find it in a way that elementary school students can understand. The forest area in the present year is the same as it was 10 years ago.

Source: (Kumakura et al., 2019)

### 3.2 Procedure

The first survey was a collective survey conducted in May 2019. It was administered in person at the university where it was also answered and collected. The data collected from university students were then compared to high school students who want to become teachers. In the second round, a survey was conducted in July 2021, with a slight modification to question 6 from “Write the method and answer” to “Write the method and answer in a way that elementary school students can understand.” This was changed to a more pedagogical and practical expression to measure the mathematical identity of prospective elementary school teachers. However, the answers to the question remained the same. Handouts containing practice exercises were distributed to the students present on campus. The survey questionnaires were distributed among the students. The students then prepared their answers at home and submitted them a week later. Both surveys were administered to students who were attending the same university.

### 3.3 Participants

The first survey was conducted with 86 third-year students at private universities studying to be elementary school teachers who agreed to participate in the research. The only arithmetic course taken by these students was arithmetic content theory in their first year. Some of the results of this first survey were reported by Kambara (2019). The second survey was conducted with 110 third-year students working toward a Bachelor of Education at the same private university who agreed to participate in the study. These were from a total of 135 students aspiring to become elementary school teachers at that private university. Private university A is a traditional school that has produced many teachers, and the number of graduates from here who find jobs as teachers is one of the highest among universities in the Kansai region of Japan.

In accordance with the code of ethics, the participants were asked to submit a consent form. All participants were provided details regarding the purpose of the research and confidentiality.

### 3.4 Statistical analysis

First, I calculated the percentage of positive responses for each question item (1–8) in the questionnaire. Each item was answered using a four-point Likert scale. Because the data were obtained on an ordinal scale and not on an interval scale, the scores

were assigned as follows: (a) Strongly agree = 6; (b) Agree = 5; (c) Don't think so = 2; and (d) Don't think so at all = 1. A multiple regression analysis using the stepwise method was then conducted. Referring to [Table 1](#), item 9, "I have confidence in teaching percentages (%)," was used as the objective variable, and items 1–8 were the explanatory variables. The implications for identity formation were then discussed.

For the survey questions on percentage comprehension, we calculated the percentage of correct answers and the average number of correct answers for the college student participants. We then compared these to the percentage of correct answers among high school students in Kumakura et al. (2019). Because we did not have access to the primary data from Kumakura et al. (2019), we did not test for differences in the means.

## 4 Results and discussion

### 4.1 Mathematical identity survey (2019)

#### 4.1.1 Results

The percentage of affirmative responses (option a or b) in the questionnaire survey is illustrated in [Table 3](#).

**Table 3.** Positive responses to the questionnaire survey (2019) (n = 86)

Item No.	Items	Percentage
1	Concept of percentages (%) finds application in subjects other than mathematics.	94.2
2	Knowledge of percentages is necessary for daily life.	97.7
3	Expressing it as a percentage (%) to see how much of the total it is.	96.5
4	Expressing the results in terms of percentage (%) shows the changes such as increase or decrease.	91.8
5	We can compare two quantities by expressing them as percentages.	82.5
6	Solving percentage problems is fun!	48.3
7	I am good at solving percentage problems.	26.7
8	I may try to solve problems in daily life by using the concept they have learned.	60.4
9	I am confident in teaching percentage (%).	15.2



The above table illustrates the percentage of affirmative responses to items 1 and 2, which pertain to the “necessity of proportions,” and items 3 to 5, which pertain to the “meaning of proportions.” Each of these items, with the exception of item 5, was found to be above 90%. The reason that the positive responses to item 5 (82.5%), which pertains to “comparing two quantities,” tended to be lower than those for items 3 (96.5%) and 4 (91.8%) could be because proportions are not used as often when comparing two quantities on a daily basis. The affirmative responses to items 6 to 8, which pertained to “attitude toward solving proportions,” were low, ranging from 20 to 60%. This indicates that the target students, those aspiring to become elementary school teachers, dislike proportions.

The percentage of affirmative responses to item 9, “I am confident in teaching percentages (%)” was also low at 15.2%. On this item, 29.1% of the respondents answered, “I don’t think so at all,” revealing that they have a strong sense of discomfort with respect to teaching percentages. A stepwise multiple regression analysis was carried out with item 9 as the objective variable and items 1–8 as explanatory variables (Table 4). The results revealed that item 7, “I am good at solving percentage problems,” had a significant positive effect on item 9, while the others had no effect.

**Table 4.** Determinants of confidence in teaching percentages

Variable	Item 9	95% Lower limit	95% Upper limit	VIF
Item 7	.404**	.213	.594	1.045
R <sup>2</sup>	.253**			

Note: \*\*p < .01, VIF: Variance Inflation Factor

#### 4.1.2 Discussion

Students who aspire to become teachers mainly encounter mathematics and arithmetic during primary school education, and they form their current “mathematical identity” through various experiences, such as meeting instructors and other students who are also pursuing the study of mathematics and arithmetic along with them. This mathematical identity is not immutable. Rather, it develops because of the motivation to become a teacher, the relearning of arithmetic and mathematics at university (by engaging in learning through activities requiring mathematical inquiry), and through relationships with others. Therefore, focusing on enabling students to feel confident about being good at solving proportion problems will lead to the formation of their

identity as instructors. Nevertheless, the knowledge gained through rote memorization—training students to repeatedly derive the correct answers to problems they learned in elementary school—will eventually be forgotten. Thus, it is necessary to improve students’ skills in understanding the essential meaning of proportions. To achieve this, it is important for instructors to devise an appropriate method to teach proportions. It is necessary to have a university education plan that integrates a unit on the concept of proportions that includes developing each hour’s instructional plan, mock lessons, and handouts for practice exercises.

## 4.2 Survey on the understanding of the “percentages” (2019)

### 4.2.1 Results

In this portion of the survey, we investigated the students’ understanding of the concept of percentages (%). The percentage of correct answers and the average number of correct answers are illustrated in [Table 5](#).

**Table 5.** Percentage (%) of correct answers

<b>Problem</b>	<b>University students (n = 86)</b>	<b>Second-year high-school students (n = 536)</b>
Q1. Second usage	100.0	95.4
Q2. Third usage	67.4	74.0
Q3. Contrastive type	80.2	69.9
Q4. pp type	80.2	73.0
Q5. pp type	66.3	61.2
Q6. p/p type	32.6	41.6
Average number of correct answers	4.3	4.2

Note: Data presented in Table 5 on second-year high-school students (high school sophomores [n = 536 students]) were derived from the results of the survey conducted by Kumakura et al. in [2019](#).

The percentage of correct answers to questions 1, 3, and 4 was more than 80%, while the percentage of correct answers to questions 2, 5, and 6 was lower than 70%. Questions 4 and 5 both refer to the same pp-type, but as Kumakura et al. ([2019](#)) stated, the percentage of correct answers to question 5 was lower than that of question 4. Pp-

type problems are written problems that can be solved by multiplying a percentage by a proportion, while p/p-type problems are written problems that can be solved by dividing a percentage by a proportion.

About 25% of the students interpreted the percentage increase/decrease “10% increase/10% decrease” in the same way as increase/decrease in quantity/number “10 person increase/10 person decrease.” In other words, a quarter of the students seemed to be unable to distinguish between percentages and physical units, such as liters (L), grams (g), or pieces (pcs). Question 6 had the lowest percentage of correct answers (32.6%), and there were many wrong answers (e.g., the percentage of deforestation is 60% or 20%). Students who answered “60%” did not correctly understand the meaning of the percentage, while those who answered “20%” did not sufficiently understand the meaning of the percentage because they arrived at the answer by simply subtracting the percentages ( $50\% - 30\% = 20\%$ ).

#### 4.2.2 Discussion

In Japan, ratios are taught in elementary school but not at the secondary level. The trend in the percentage of correct answers of university students and high school students, who differed in terms of age, region, and academic distribution, was highly similar. It seems unlikely that the understanding of proportions will improve naturally when these elementary school students become adults with more life experience. Additionally, the difficulty level of the problems increased with respect to the question order: second usage [(Q1)] → contrastive type [(Q3)], pp-type (no change in standard quantity) [(Q4)] → third usage [Q2], pp-type (with change in standard quantity) [Q5] → pp-type [(Q6)]. This indicated that students might not naturally deepen their understanding as they progressed through the grades. Therefore, it is necessary to ensure that students understand that proportion problems are of varying difficulty levels and can be used for teaching “proportions.”

### 4.3 The understanding of “question 6” (2021)

#### 4.3.1 Results

Among the 110 students, 93 (84.5%) answered question 6 correctly (i.e., 40% of the deforested area). [Table 6](#) depicts the qualitative classification of the students’ answers, focusing on their expressions.

**Table 6.** Evaluation criteria for categorizing students (n = 110)

<b>Evaluation criteria</b>	<b>Number of respondents</b>
Type A: The meaning of proportions is expressed correctly and described without logical leaps.	15 (13.6%)
Type B: The meaning of proportion is expressed correctly and described almost logically. Some leaps are made, with some examples not being close to reality.	12 (10.9%)
Type C: There is some problem understanding the meaning of proportions, and the explanation is insufficient.	26 (23.6%)
Type D: There is a problem understanding the meaning of proportions, and there are many inadequate explanations.	40 (36.4%)
Type E: There is a problem understanding the meaning of proportions, and the answer is wrong.	17 (6.4%)

A total of 27 students (24.5%) were included in Types A and B, while 66 students were in Types C and D, which is not a small number. This indicates that many students could answer the question correctly but could not provide an appropriate explanation.

A typical example of a Type A response (13.6%) is as follows:

Let the total area of town A be 1. Ten years ago, the forest area was 50% of the total area, and thus it was 0.5. This year's forest area is 0.3 because it is 30% of the total area of Town A. Based on the area of the forest 10 years ago, we know that this year's forest area has decreased by  $0.5 - 0.3 = 0.2$ . Based on the area of the forest 10 years ago, we know that the forest area has decreased by  $0.2 / 0.5 = 0.4$ , or 40%. The forest area decreased by 40%.

Students whose responses were categorized as Type A were able to state reference quantities in their explanations, for example, "Let the total area of town A be 1."

A typical example of a Type B response (10.9%) is as follows:

Let the total area of town A be 100 m<sup>2</sup>. Ten years ago, 50% of the total area was 50 m<sup>2</sup>. This year, it is 30% which is equal to 30 m<sup>2</sup>. Based on the area of the forest 10 years ago, the area of the forest this year is  $50 - 30 = 20$ , indicating a decrease of 20 m<sup>2</sup>. If we express 20 m<sup>2</sup> as a percentage, we obtain  $20 / 50 \times 100 = 40$ , that is, a 40% decrease.

Thus, the 11 members of Type B proceeded to discuss the total area of Town A as 100 m<sup>2</sup>. The remaining members assumed that the total area of Town A was 200 m<sup>2</sup>. This indicates that these students described the area of a town as being about the same

size as a classroom. Additionally, the description ignored the fact that the discussion of the assumed area was not generally applicable.

A typical example of a Type C response (23.6%) is as follows: “Based on the forest area 10 years ago, this year’s forest area is 30% of that,  $0.3/0.5 = 0.6$ , which is 60%, and  $100 - 60 = 40$ , which is 40%.” The example demonstrates that the standard quantity is misrepresented; the standard quantity value is unclear, and the calculation procedure is not explained. Most of the responses in Category C ask for the solution as “percentage - percentage = amount of decrease.” However, it is not clear whether the fact that the reference quantity A has not changed is implicit or due to a lack of understanding; in any case, it is not expressed correctly.

A typical example of a Type D response (36.4%) is as follows: “Let 50% be 1. If we replace 50% with 100%, then 30% is 60% of 50%;  $100-60=40$ , thus a 40% decrease.” Thus, the explanations in Type D do not adequately state the relationship between proportions, reference quantities, or comparison quantities. Additionally, students do not express themselves with the awareness that they are explaining to elementary school students. A lack of explanatory language is another characteristic of Type D. Type E (15.5%) refers to those who answered 60%, 20%, or 50% as their answers. There were nine, seven, and one student in this category, respectively.

#### 4.3.2 Discussion

As described above, p/p-type problems are not easy, even for students who wish to become elementary school teachers. Even those students who answered correctly had difficulty providing appropriate explanations and presenting the right amount of information for the situation. In arithmetic and mathematics learning, the ability to use proper mathematical language and have an appropriate sense of quantity are important qualities and competencies that we want children to acquire. Deepening the understanding of ratios among students aiming to become instructors and fostering the ability to explain the use of mathematical expressions and sense of quantity are issues that need to be addressed. Since the ability to provide correct explanations is inextricably linked to understanding mathematics, we must provide guidance for students to help them understand the content and refine their mathematical expressions. We believe that such guidance will help students develop a sound mathematical identity. In particular, proportion is an important topic because it is a significant concept when considering quantities per unit and in the functional domain. Additionally, “proportions” involve a mathematical concept that most students find difficult; thus,

using this topic as a reference for refining students' understanding and methods of expression might be an effective way of helping them form a sound mathematical identity.

## 5 Conclusion

The purpose of this study was to clarify and investigate the level of understanding of the concept of “proportions” among students who want to become elementary school teachers and are undergoing training at a private university and obtain recommendations for the development of students' mathematical identities.

Of the survey participants, 15.2% responded positively to the question about teaching proportions. When it comes to teaching proportions, it was found that many students have a negative mathematical identity as instructors. Therefore, we investigated the determinants of “confidence in teaching proportions.” A stepwise multiple regression analysis was carried out with item 9, “I am confident in teaching percentages (%)” as the objective variable and items 1–8 as explanatory variables (Table 4). The results revealed that item 7, “I am good at solving percentage problems,” had a significant positive effect on item 9. In other words, it was found that being good at solving proportions could lead to a positive mathematical identity as a teacher.

Next, because it is necessary to understand the status of students' understanding of proportions, I examined the status of proportion-related problem solving and found the following: Examining mathematical identity and understanding proportions among college students who wanted to become elementary school teachers at University A revealed that the difficulty level of the numerical problems increases in the order of the second usage → contrast type, pp-type (no change in standard quantity), third usage, pp-type (with change in standard quantity), and p/p-type. Particularly, students had problems understanding the third usage and cases where the standard quantity was unknown (p/p-type). As such, students must acquire knowledge about these problems. This is similar to the results obtained by Kumakura et al. (2019) for high school students, suggesting the need for universities to provide students with opportunities to relearn proportions. In particular, for p/p-type problems, many students could not explain the correct reasoning, even when they could derive the correct answer. This suggests that there is a need for students to learn how to explain the problem-solving process. Because the ability to provide correct explanations is inextricably linked to understanding mathematics, it is essential that we provide guidance to students on the content and how to refine their expression. We



believe that such guidance will help students develop a positive mathematical identity. Deepening the understanding of ratios among students aiming to become instructors and fostering their sense of quantity and ability to explain the use of mathematical expressions are also issues that need to be addressed.

This survey was limited to students at a private university in Japan who wanted to become elementary school teachers. In the future, the survey should be conducted at other universities worldwide to assess if the results differ depending on the differences in curricula in each country. Additionally, considering that the descriptive survey for question 6 was a collection and distribution survey, it is expected that the results would have been lower than the present results if a group survey had been conducted.

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# Student teachers' common content knowledge for solving routine fraction tasks

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This study focuses on the knowledge base that Swedish elementary student teachers demonstrate in their solutions for six routine fraction tasks. The paper investigates the student teachers' common content knowledge of fractions and discusses the implications of the findings. Fraction knowledge that student teachers bring to teacher education has been rarely investigated in the Swedish context. Thus, this study broadens the international view in the field and gives an opportunity to see some worldwide similarities as well as national challenges in student teachers' fraction knowledge. The findings in this study reveal uncertainty and wide differences between the student teachers when solving fraction tasks that they were already familiar with; two of the 59 participants solved correctly all tasks, whereas some of them gave only one or not any correct answer. Moreover, the data indicate general limitations in the participants' basic knowledge in mathematics. For example, many of them make errors in using mathematical symbol writing and different representation forms, and they do not recognize unreasonable answers and incorrect statements. Some participants also seemed to guess at an algorithm to use when they did not remember or understand the correct solution method.

Keywords: common content knowledge, elementary school, fractions, student teacher, teacher education

## 1 Introduction

Teaching and learning of fractions has shown to be a challenging area in mathematics (e.g., Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Löwing, 2016; Ma, 2010; Newton, 2008). As Lamon (2007, p. 629) expresses, fractions like ratios and proportions are “the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science.” Nevertheless, fractions are an essential part of school mathematics and an important part in the development of algebra and proportional reasoning. Elementary school students' knowledge of fractions and division can even predict their algebraic skills and performance in mathematics several years later (Siegler et al., 2012).

A deep understanding of rational numbers requires knowledge of different fraction interpretations such as the operator model and linear models (see e.g.,

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Kieren, 1993; Lamon, 2007, 2020). However, student teachers seem to favor the part-whole model that has traditionally been connected to fractions and taught in elementary schools, and they struggle with other fraction interpretations (Lamon, 2020; Olanoff et al., 2014). Developing skills with fractions also requires the ability to perform fraction operations and to build up some degree of fraction sense. According to Lamon,

This means that students should develop an intuition that helps them make appropriate connection, determine size, order, and equivalence, and judge whether answers are or are not reasonable. Such fluid and flexible thinking is just as important for teachers who need to distinguish appropriate student strategies from those based on faulty reasoning. (Lamon, 2020, p. 143)

In Sweden, the national curriculum for the compulsory school states the core content related to fractions first as parts of a whole and as parts of whole numbers, which should be compared and named as simple fractions in grades 1-3 (Skolverket, 2011). Further, in grades 4-6, the knowledge requirements include an understanding of rational numbers in fraction, decimal and percentage form. The main calculation methods for fractions are included in the curriculum for grades 7-9. Even though efforts have been made to improve learning results in mathematics, studies show that Swedish elementary school students still have deficiencies in fulfilling the above knowledge requirements (Löwing, 2016; Skolverket, 2016, 2019). Therefore, it is also important to focus on student teachers and to study their knowledge of fractions thoroughly.

Previous studies (e.g., Ma, 2010; Tirosh et al., 1998; Zhou et al., 2006) have shown the important role of teacher education in developing student teachers' fraction knowledge and the need for further research and international comparisons in this topic (Olanoff et al., 2014). The present study is a part of a more comprehensive research project that seeks to respond the research needs in this field by expanding the view to the Swedish teacher education context. The aim of this paper is to investigate Swedish elementary student teachers' common content knowledge (CCK) of fractions by analyzing errors and difficulties in their solutions for routine fraction tasks. The research question of this study is:

How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks?

## 2 Previous research on student teachers' fraction knowledge

A number of studies investigating different aspects of student teachers' fraction knowledge have been published in mathematics education research. Olanoff et al. (2014) present a summary of 43 research articles focusing on student teachers' mathematical content knowledge in the area of fractions. These studies conducted, e.g., in Australia, Taiwan, Turkey and in the USA between the years 1989 and 2013, show that student teachers' fraction knowledge is relatively strong in performing fraction procedures. However, when including all basic operations of arithmetic and using basic fraction tasks that can be found in elementary school mathematics textbooks some studies also show limitations in student teachers' knowledge of fraction operations (e.g., Newton, 2008; Young & Zientek, 2011).

For example, Newton (2008) identified several error patterns when studying elementary student teachers' knowledge of routine fraction tasks in the USA. For addition, and especially when the denominators were different, the most common error was adding across numerators and denominators. In the subtraction of fractions, student teachers had difficulties changing forms, they subtracted across and left blank. In multiplication, they made whole-number errors with mixed numbers, cross-multiplied fractions instead of multiplying across, kept the common denominator in the answer, added numerators or denominators, and made errors in changing forms as well. Student teachers in Newton's study were most uncertain about dividing fractions, and even more error patterns were found for that operation: (a) finding a common denominator and keeping it in the product, (b) leaving blank, (c) reciprocals, (d) flipping the dividend instead of the divisor, (e) making mistakes with whole number facts, (f) cross-dividing or cancelling, and (g) adding or subtracting numerators or denominators. Newton (2008) concluded that the most common error in the operations with the routine fraction tasks was keeping the denominator the same even though it was not suitable.

A few years later, Young and Zientek (2011) showed that student teachers' competence vary by fraction operation; division and multiplication are the most difficult operations for student teachers. Moreover, student teachers' knowledge of fraction operations was partly rule-based and, for example, they tended to overgeneralize the rule of converting fractions to have like denominators for multiplication as well. Many of the student teachers' error patterns seemed to be based on incorrect memories of algorithms they had learned before which led them to inappropriate use of procedures; in some tasks they used correct procedures and in

some other tasks with the same operation they chose the incorrect ones. Thus, Young and Zientek (2011) concluded that student teachers in their study were not able to accurately judge their abilities to correctly perform the fraction operations.

Previous research have also reported on student teachers' difficulties understanding the meanings behind fraction procedures and why the procedures work (e.g., Ma, 2010; Marchionda, 2006; Olanoff et al., 2014). Tirosh (2000) concludes that many student teachers in Israel are not capable of explaining the fraction division procedure even though they are able to use it. Similarly, the American final-year student teacher in Borko et al.'s study (1992) showed a weak understanding of both multiplication and division of fractions at the end of her teaching practice after completed a mathematics methods course; her knowledge of fraction division was based on a rote understanding of the invert-and-multiply algorithm and she lacked any knowledge of other representations such as visual representations of fractions she could use to demonstrate the division solution. Moreover, student teachers seem to lack flexibility in moving away from procedures and using fraction number sense, for example, when converting a fraction to a decimal (Muir & Livy, 2012; Olanoff et al., 2014). This may be one reason many student teachers have difficulties solving fraction story problems and creating their own fraction word problems (e.g., Ball, 1990; Tirosh, 2000; Toluk-Uçar, 2009).

Researchers in previous studies have also concluded that the relationship between student teachers' conceptual and procedural knowledge of fraction operations is weak, and that their fraction knowledge reflects the misconceptions that children have when working with fractions (e.g., Lin et al., 2013; Van Steenbrugge et al., 2014; Young & Zientek, 2011). Similar to children, many student teachers make errors based on prior knowledge of whole numbers, and when misapplying algorithms, especially the multiplication algorithm, student teachers' errors can also relate to their prior knowledge of fractions, e.g., to cross-multiplying which can be used when comparing fractions (Newton, 2008).

Student teachers are assumed to have a certain level of competence in using fractions when they are admitted to teacher education. However, Van Steenbrugge et al. (2014) concluded that one reason Flemish student teachers perform at a low level with fractions is the limited time spent on fractions in teacher education. Teacher education does not seem to have an impact on student teachers' common content knowledge of fractions, which reveals a need to develop mathematics teaching in this area (Van Steenbrugge et al., 2014).



Even though the multiple challenges related to the teaching and learning of fractions are widely recognized in many international studies as shown in the examples above, there seem to be few recent studies focusing on student teachers' fraction knowledge in the Nordic countries. One such study focuses on Icelandic student teachers' mathematical content knowledge showing that they have considerable difficulty with fractions; their knowledge is procedural and relates to "standard algorithms" learned in elementary school (Jóhannsdóttir & Gíslandóttir, 2014). A study of Norwegian student teachers (Jakobsen et al., 2014) shows that they have difficulties when solving fraction word problems; the student teachers seem to lack familiarity with mathematical notions of fractions, and they have difficulties interpreting elementary students' solutions and giving sense to fraction solutions different from their own. Furthermore, a study conducted in Finland indicates that a large number of those applying for teacher education have challenges in solving fraction algorithms (Häkkinen et al., 2011). As stated in many previous studies in the field, researchers in these Nordic studies as well highlight teacher educators' responsibility in ensuring the quality of student teachers' fraction knowledge and the need for further research in this area. The present study contributes to the field by taking the topic to Swedish teacher education and presenting an analysis of student teachers' CCK of fractions in a Swedish context. This gives an opportunity to see some worldwide similarities and national challenges in student teachers' fraction knowledge.

### 3 Theoretical framework

Over the last few decades, an increasing research interest has been given to subject matter knowledge as an important part of teaching (e.g., Shulman, 1986). In his original work, Shulman (1986) suggests three categories of teacher knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Subject matter knowledge includes not only the knowledge of the content of a subject area but also knowledge of substantive and syntactic structures. By these, Shulman refers to the varying ways the basic concepts, principles and facts of a discipline are organized and identifies the legitimate rules in that domain. Successful teaching requires also pedagogical content knowledge, what Shulman (1986) calls "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

In mathematics, there has been a lack of agreement about definitions, language, and basic concepts within teaching-specific mathematical knowledge (Hoover et al., 2016). Ernest states that

the teacher's knowledge of mathematics is a complex conceptual structure which is characterized by a number of factors, including its extent and depth; its structure and unifying concepts; knowledge of procedures and strategies; links with other subjects; knowledge about mathematics as a whole and its history. (Ernest, 1989, p. 16)

Many studies concerning student teachers' knowledge base have focused on the differences between their conceptual and procedural knowledge (e.g., Lin et al., 2013; Marchionda, 2006). Conceptual knowledge is knowledge that is rich in relations (Hiebert & Lefevre, 1986). When it comes to fractions, it includes the understanding of the definition of fractions and other relevant number sets, fundamental facts about these numbers, and how the essential facts are related in the context of fraction tasks. Procedural knowledge about fractions concerns computational skills that are needed for solving fraction tasks and familiarity with the proper ways to denote fractions and their operations, for example, how to use appropriate rules and notations for the division of fractions (Hiebert & Lefevre, 1986). Maciejewski and Star (2016) conclude that flexible procedural knowledge is a key skill, which can be a way to improve students' conceptual knowledge as well. However, as Newton (2008) states, "dichotomizing mathematical knowledge into procedures and concepts does not account for its complexity" (p. 1105).

Even though teacher knowledge base has been regarded as an essential part of effective teaching, scholars have argued whether and how it contributes to students' learning. Thus, several studies have been conducted to examine the extent to which, for example, the mathematical knowledge for teaching framework (MKT) relates to learning (e.g., Charalambous et al., 2020). When analyzing the mathematical demands of teaching, Ball et al. (2008) identified the mathematical knowledge that is needed for teachers to effectively perform their work. They present the MKT framework based on Shulman's (1986) knowledge categories by using domains of subject matter knowledge and pedagogical content knowledge, and suggest that many teaching tasks included in the subject matter knowledge domain require mathematical knowledge that is not dependent on the content in the pedagogical domain.

This study focuses on the common content knowledge (CCK) category of the subject matter knowledge domain. Ball et al. (2008) define CCK “as the mathematical knowledge known in common with others who know and use mathematics” (p. 403). This knowledge and skill are used in a wide variety of settings in day-to-day work, and is thus not unique to teaching. CCK can be regarded as a basic competence in mathematics since it includes, e.g., performing calculations correctly, carrying out mathematical procedures, recognizing wrong answers, and using definitions, terms and notations correctly as well as understanding fractions (Ball et al., 2008). CCK covers mathematical tasks and questions that can be answered by anyone with a general knowledge of mathematics.

A robust CCK is a requirement for specialized content knowledge (SCK), which contains mathematical knowledge and skills that are used in teaching settings and are typically not needed for purposes other than teaching. As Ma (2010) concludes, “in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it” (p. 71). SCK includes abilities like explaining why common denominators are used when adding fractions and what is the procedure behind the invert-and-multiply algorithm in dividing fractions or determining whether a nonstandard approach would work in general to solve a given problem (Ball et al., 2008). In other words, this is knowledge of how to make mathematics understandable to students. However, in some cases it can be difficult to differ CCK from SCK. For example, detailed knowledge of different fraction representations such as symbolic and pictorial representations can be regarded as specialized knowledge, but it can also be common knowledge for others in their daily work (Ball et al., 2008).

Based on Ball et al.’s (2008) description of CCK, the present study investigates elementary student teachers’ CCK by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. The concept error is chosen for this study instead of, e.g., misconception or misunderstanding, and its definition for this study is presented later in this paper. The concept difficulty is also used since it was assumed that not all findings in the analyzed fraction solutions could be categorized as obvious errors. However, this paper does not intend to explain why specific errors appear. As Radatz (1979) states, “errors in the learning of mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is such a close interaction among causes” (p. 164).

## 4 Methodology

### 4.1 Participants

The participants in this study were 59 university students in Swedish elementary school teacher programs, which are meant for to prepare teachers for the preschool class and grades 1-6 of compulsory school. Most of the participants were in the third academic year of their four-year programs, and they had already passed their first mathematics course in teacher education. One of the key aims of this mandatory course is to deepen student teachers' mathematical knowledge and strengthen their computational skills. During the first mathematics course, fraction content that is studied before entering teacher education and included in the curriculum for the compulsory school, e.g., calculating with fractions by using all operations, simplifying, reducing and extending fractions, and converting fractions to decimal, percent and mixed number forms, is recalled and repeated with all student teachers. At the time of the present study, the participating student teachers were starting their second mathematics course, which had a focus on the didactics of mathematics.

### 4.2 Data collection

Data for this study were collected by using a printed questionnaire. The voluntary participants were given 90 minutes to answer it before the first lecture of their mathematics didactics course at the university campus. They were asked for some background information (part 4 in the questionnaire), to write about the concept of fraction (part 1), and to describe how they might teach a fraction addition task to elementary school students (part 2). This paper focuses on six routine fraction tasks that were included in the questionnaire as well (part 3, see Appendix A). The instruction for the tasks was presented as follows: 'Calculating with fractions. Solve the following tasks as well as you can without using a calculator. Show all the steps you use.' With the instruction 'show all the steps', the participants were indirectly guided to show their fraction knowledge using mathematical algorithms, which they had been repeating in the previous mathematics course in teacher education and which can be regarded as CCK for mathematics teachers. It was also possible to use other representations such as pictures or decimal forms since the instruction was written: 'Solve the following tasks as well as you can'. Detailed knowledge of fractions and their correspondence to different representations is also knowledge that

mathematics teachers need in their daily work (Ball et al., 2008).

The six fraction tasks used in this study were based on similar tasks that can be found in Swedish mathematics books and support materials for grades 4-6 mathematics. All four operations, i.e. addition, subtraction, multiplication and division, were included in the tasks with different types of fraction content: (a) addition with common denominators, (b) addition with different denominators, (c) subtraction with different denominators, (d) subtraction with a whole number, (e) multiplication with different denominators, and (f) division by a whole number. The participating student teachers were already familiar with this kind of tasks, and the tasks were defined as routine tasks since the operations were written without any context (c.f. Newton, 2008).

### 4.3 Data analysis

In this study, elements from Radatz's (1979) information-processing classification were used to categorize the errors in the participants' solutions. Three error types were of interest in the analyzed routine tasks: errors that are due to (1) lacking knowledge of prerequisite skills, facts, and concepts, (2) incorrect associations or inflexibility in thinking, and (3) application of irrelevant rules or strategies. Radatz (1979) states that category (1) "includes all deficits in the content- and problem-specific knowledge necessary for the successful performance of a mathematical task" (pp. 165-166), and he continues by elaborating "Deficits in basic prerequisites include ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of necessary concepts and symbols" (p. 166). The error type (2) includes negative transfer from similar tasks even though the conditions for the tasks are different. In the last category, the errors are mainly based on successful experiences when applying comparable rules or strategies in other content areas. However, making a clear distinction between those error types mentioned above is often difficult because many of the causes interact during the learning process (Radatz, 1979).

When analyzing the participants' solutions, the answers were first coded as correct or incorrect. As a correct answer, it was assumed in this study that the answer was converted to a mixed number when possible or that it was presented in the simplest fraction form. This decision was based on the instructions and examination of the previous mathematics course, which the participants had passed in their teacher education. In Sweden, simplifying and extending fractions are considered to be

prerequisite skills for the addition and subtraction of fractions (Löwing, 2016). Thus, giving the answers for fraction tasks in the simplest fraction form or as a mixed number is also encouraged in Swedish compulsory school mathematics books, and it is often expected that the answers are primary given as a fraction and not as a decimal or percent, which might be mathematically correct as well. However, providing an answer in these forms was not mentioned in the instruction since in another part of the questionnaire it was examined whether the participants were able to provide these fraction-related concepts themselves.

After the first round of coding, a qualitative analysis focusing on the solution methods was conducted. It was investigated whether there were solution methods used other than mathematical symbol representations and what kind of errors were included in the solutions. However, using other methods than mathematical algorithms was not classified as an error. Following Young and Zientek (2011), errors were defined as technical and procedural errors, where the latter consist of obvious errors in using fraction operations. This was in the cases where the participants were misusing the procedures, for example, adding across numerators and denominators in addition. This refers to Radatz's (1979) first error type. Also, if their methods seemed inefficient or misleading when used in teaching settings, and if there seemed to be a lack of number sense or negative transfer from similar tasks in the solutions, the operations were classified as including errors in this study. For example, this was done in the cases where the participants were using unnecessary long solution methods or big common denominators, or they were using common denominators when unnecessary. This classification has a connection to Radatz's error type (2) presented above.

Before deciding on the final error categories, the errors were coded several times to ensure the reliability of the coding. The rating of the errors was also discussed with an additional researcher and after that, the primary errors were coded by using symbols E1, E2, E3 etc. (see Appendix B). The errors were categorized altogether as seven error types. Three of them are related to fraction operations (procedural errors): errors in addition or in subtraction (E5), errors in multiplication (E6), and errors in division (E7). These categories include several subtypes of errors that were made by individual or multiple students.

Technical errors in this study are related to presenting the answer (E1), mathematical writing (E2), mathematical facts (E3), and leaving the task blank in the research questionnaire (E4). These errors include also solutions that can be regarded



as correct in contexts other than this study. For example, E1 category consists of five subcategories that describe the solutions, which were counted to be incorrect in the context of this study even though the answers may otherwise be mathematically correct, i.e. presenting the answer as decimal or not as a simplified fraction form. E2 includes partial computation and missing solution steps as well as illogical mathematical symbol writing, and E3 consists of minor errors in calculation. Despite mathematical writing errors in the procedures, the participants' solutions have been counted as correct in the analysis if they produced a correct answer for the fraction task.

The findings of the study were analyzed in terms of the number or percentage of participants who successfully performed the fraction tasks by providing correct answers or those who made errors in their solutions. Otherwise, the main data analysis was based on a qualitative description of the student teachers' solutions for the tasks. When analyzing the solutions, the anonymous participants were given number codes according to the order their questionnaires were analyzed. These number codes are used as references in the figures presented in the next section.

## 5 Results

The research question of the study 'How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks' will be answered next. This section begins by describing the participants' fraction solutions in general; their errors and difficulties with the different tasks will then be described in more detail.

### 5.1 On student teachers' solutions for the routine fraction tasks

**Table 1** shows the number of student teachers giving correct answers for the fraction tasks, using pictorial representations and making the most common technical errors E1, E2 and E4. As can be seen in **Table 1**, there is a wide difference between the student teachers when solving the routine fraction tasks. Two of the 59 participants gave correct answers to all six tasks, whereas on the other end of the spectrum, there were participants that gave only one or not any correct answer. The participants with all correct answers used mathematical symbol representations and wrote their solution steps in the algorithms in such a way that it was easy to follow the procedures they used. The participants with the least correct answers made errors with all operations,

and they had difficulties in simplifying the fractions and converting them to mixed numbers. Only one of these student teachers seemed to demonstrate knowledge in using the different algorithms and writing the mathematical steps; otherwise, the participants with the least correct answers did not seem to notice the errors they made with the operations.

**Table 1.** A summary of the participants' solutions for the fraction tasks

Number of correct answers	Number of participants	Number of participants using pictures	Number of participants making errors in presenting the answer (E1)	Number of participants making mathematical writing errors (E2)	Number of participants leaving blank (E4)
6 (all correct)	2	0	0	0	0
5	11	2	4	8	2
4	17	3	8	13	3
3	16	2	14	10	3
2	5	0	3	3	4
1	4	0	4	3	1
0	4	0	2	1	3
<b>Total</b>	<b>59</b>	<b>7</b>	<b>35</b>	<b>38</b>	<b>16</b>

In general, the participating student teachers did not show a robust CCK in presenting mathematical algorithms and solutions steps. Almost a half of the participants failed to follow the instruction to show all their solution steps at least with one of the tasks. This may indicate that they had difficulties in mathematical symbol writing or that they did not notice where or how to write more details in their solutions. This was most common in the case of division where only six participants presented a logical mathematical solution by using fractions. For example, the step showing how to do the change to common denominators is missing in the next solution even though the mathematical writing is done correctly and the right answer is found:  $\frac{4}{5} + \frac{2}{3} = \frac{12}{15} + \frac{10}{15} = \frac{22}{15} = 1\frac{7}{15}$ . Furthermore, the participants using pictorial representations did not present any steps with their solutions. However, they provided more often the correct answers for the tasks than those who used mathematical algorithms incorrectly in their solutions.

Moreover, the participants' CCK in using different representations in their fraction solutions seemed limited. Some participants used decimals but they made errors in giving correct answers; one of them used decimals for all the tasks without ending to any correct answer. Pictorial representations were used most often to solve the

division task. The multiplication task  $\frac{3}{4} \cdot \frac{2}{5}$  was not solved with pictures, which may indicate that the multiplication procedure is more challenging to present with pictures than the other fraction operations in the analyzed tasks. Also, it seemed that the participants used pictures for the tasks that were easier to visualize with pie charts; for example, not for the addition task with different denominators 5 and 3. Moreover, when the participants used two separate circles for subtraction, the circles (pie charts) in their solutions seemed to represent the fractions rather than the subtraction procedure (see [Figure 1](#)). In the case of addition, two circles can easier be used to illustrate the procedure as well (see [Figure 2](#)). To summarize, it seemed that the participating student teachers' CCK knowledge for using pictorial fraction representations to demonstrate solution procedures was limited.


$$\frac{3}{4} - \frac{1}{2} =$$


Figure 1. A subtraction solution with pie charts (participant 16)

$$\frac{2}{3} + \frac{2}{3} =$$


Figure 2. An addition procedure illustrated with pie charts (participant 22)

Many participants also made different kinds of obvious E3 errors in their solutions. These errors in mathematical facts did not seem to be directly related to fractions but were rather simple mistakes in calculation, like  $12+10=24$  and  $3 \cdot 3=6$ . Some participants also made multiple error types in their solutions, e.g., they used illogical mathematical writing for a wrong solution method and made calculation mistakes as well (see [Figure 3](#)).

$$\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 5}{4 \cdot 5} \quad \frac{15}{20} \cdot \frac{8}{20} \quad \frac{15}{20} \cdot 84 \quad \frac{120}{20} = \frac{120/10}{20/10} = \frac{12/2}{2/2} \frac{6}{1} = 6$$

$$\frac{2 \cdot 4}{5 \cdot 4} \quad \frac{8}{20}$$

Figure 3. A solution with multiple errors (participant 41)

The most common technical error types were E1, E2 and E4 (see [Table 1](#)). Most of the participants who had difficulties in mathematical writing (E2) did not use mathematical notations correctly throughout their solutions; many of them used the equal sign incorrectly presenting their solutions often as separate calculations and ignoring whether the equal sign was written between the solution steps or not. The thinking model behind these solutions can often be understood, but mathematically, this kind of partial writing results in illogical statements (see [Figures 3 and 4](#)).

$$1 - \frac{2}{6} = 1 = \frac{6}{6} \quad \frac{6}{6} - \frac{2}{6} = \frac{4}{6} \quad \frac{4}{6} = \frac{2}{3}$$

Figure 4. A solution with illogical writing (participant 27)

Several participants also used the division sign incorrectly and, in particular, they seemed to have difficulties in making a distinction between dividing and simplifying the fractions with their notations (see [Figure 5](#)).

$$\frac{3}{4} \cdot \frac{2}{5} = \frac{6}{20} / 2 = \frac{3}{10} \quad \frac{3}{4} \cdot \frac{2}{5} = \frac{6}{20} / 2 = \frac{3}{10}$$

Figure 5. Examples of errors in using the division sign (participants 47 and 48)

It seems that the participants who provided the solutions above were using division while meaning to simplify the fraction  $\frac{6}{20}$ , which should have led to an answer that was different from the one they provided. However, some participants were able to use the mathematical notations correctly, writing, for example:  $\frac{3}{4} \cdot \frac{2}{5} = \frac{6/2}{20/2} = \frac{3}{10}$ .

More than half of the participating student teachers made errors concerning the proper form for the answer (E1), and their uncertainty and illogical use of different fraction forms could be found in many solutions: in some tasks they provided the answer as a simplified fraction or a mixed number while in other similar cases, they did not. If neglecting these technical E1 errors, the total number for correct answers in the tasks would have been greater; still, it would not have led to all answers correct in any of these routine fraction tasks, and only seven participants would have correctly solved all tasks.

Several participants also left at least one of the tasks blank. This leaving blank error (E4) was made in all types of the fraction tasks except addition with common denominators, and it was most common for multiplication and division, which were both left blank by 10 students. Leaving blank may indicate uncertainty in using the procedures when the participants did not remember the correct algorithms.

## 5.2 Student teachers' errors and difficulties with the routine fraction tasks

The number of participants giving correct answers and making different error types E1-E7 in the analyzed six fraction tasks are summarized in Appendix B. An analysis of their errors and difficulties with the fraction tasks is presented below in the same order as the tasks existed in the questionnaire.

**Addition with common denominators:**  $\frac{2}{3} + \frac{2}{3}$ . Altogether, 42 participants (71%) gave the correct mixed number answer for this task. Five of them showed detailed steps in their solutions, writing, for example,  $\frac{2}{3} + \frac{2}{3} = \frac{2+2}{3} = \frac{4}{3} = 1\frac{1}{3}$ . Some participants may have perceived this task so simple that there was no need to show detailed solution steps, and four participants used a pictorial representation (circles or rectangles) as a method to find the correct answer.

Most errors here were technical E1 errors. Six participants gave the answer as an improper fraction  $\frac{4}{3}$  instead of converting it to a mixed number, and one participant gave the answer as a decimal, i.e. 1.33. Four student teachers seemed uncertain and wrote their mixed number answers within parentheses or as an unfinished answer in two parts  $1 + \frac{1}{3}$  or they gave even two alternative answers,  $\frac{4}{3}$  or  $1 + \frac{1}{3}$ . Moreover, nine participants made a procedural E5 error by adding across the numerators and denominators. After adding incorrectly, three of them also simplified the fraction  $\frac{4}{6}$  to  $\frac{2}{3}$  without noticing that this was not a reasonable answer when adding  $\frac{2}{3} + \frac{2}{3}$ .

**Addition with different denominators:**  $\frac{4}{5} + \frac{2}{3}$ . Compared with the first addition task, a smaller amount of participants, 37 of 59 (62%), performed correctly this task. Those who had difficulties in the previous task made similar E1 errors in presenting the answer here as well. One participant converted his/her improper fraction solution again to a decimal number (1.466). However, all these participants as well as those with the answer in the correct mixed number form, showed their mathematical solution steps: they found the common denominator for the given fractions and used a proper solution method. Some participants made minor

computational errors (E3), and two student teachers left this task blank (E4). Interestingly, the other of them solved correctly the previous task (addition with same denominators) and the next one (subtraction with different denominators). Thus, it seemed that he/she was uncertain about the role and use of denominators in these fraction tasks.

Technical error E2 was the most common error type here since several participants used incorrect mathematical notations, and had partial computations or missing solution steps. However, there were even more procedural E5 subtype errors in the addition operation. Ten participants used a total of seven different faulty methods for the addition operation, which led to as many different incorrect answers. Three of these student teachers found the common denominator 15, but they multiplied only the denominators, adding the fractions as follows:  $\frac{4}{15} + \frac{2}{15} = \frac{6}{15}$ . One participant used an unnecessarily large common denominator, 30, instead of 15. Even though mathematically correct, this method seemed inefficient and it can also be interpreted as a lack of number sense. Two participants added across numerators and denominators; the other of them did this even though he/she did not add the like denominators in the first addition task. Four participants used varying multiplicative methods, for example, they cross-multiplied or multiplied across the numerators and denominators. One student teacher cross-added twice and ended up with the solution presented in [Figure 6](#). In the solution, the participant added across the common denominators, which he/she did with the previous addition task as well.

$$\frac{4}{5} + \frac{2}{3} = \frac{4+3}{5+2} + \frac{2+5}{3+4} = \frac{7}{7} + \frac{7}{7} = \frac{14}{14}$$

Figure 6. An incorrect solution for addition (participant 40)

One participant seemed to demonstrate uncertainty when presenting two alternative solutions. The other solution procedure and the resulting answer  $1\frac{7}{15}$  were correct, but he/she had marked the following method as the correct one:  $\frac{4}{5} + \frac{2}{3} = \frac{4+3}{5+3} + \frac{2+5}{3+5} = \frac{7}{8} + \frac{7}{8} = \frac{14}{8} = 1\frac{6}{8}$ . In general, the participants who made errors with their addition solutions did not seem to notice that their answers were unreasonable. For example, when adding  $\frac{4}{5} + \frac{2}{3}$ , it is not possible to get  $\frac{1}{5}$  as an answer because it is smaller than  $\frac{4}{5}$ . The number of different incorrect solution methods in this task may indicate that



when the participants did not remember or understand the operation procedure they seemed to guess an algorithm to use for the solution.

**Subtraction with different denominators:**  $\frac{3}{4} - \frac{1}{2}$ . This task was correctly performed by 47 participants (80%); four of them used pie charts to present their solutions while the others showed their solutions with some mathematical steps. One participant converted the fractions first to percent and then after calculating the answer it was converted to the correct fraction form, which was an example of using different representation forms to find the solution. Moreover, two participants used decimals, and one of them arrived at a right decimal form answer. Three participants left this task blank.

Several participants made technical E2 writing errors also with this task. Procedural E5 errors were made as well, and the most common of them was the use of unnecessarily large common denominators: seventeen participants multiplied both fractions in order to get 8 as the common denominator. This may indicate a lack of number sense related to whole numbers or a poor understanding of the subtraction operation since it was not necessary to multiply both fractions since the denominators were 4 and 2. Moreover, one participant found the common denominator 8 but kept multiplying the numerators following the same logic as he/she did in the latter addition task as well. Two student teachers who added across in addition used a similar method here as well. Thus, they subtracted across the numerators and denominators and wrote the problem out as:  $\frac{3}{4} - \frac{1}{2} = \frac{2}{2} = 1$ . Here, again, it can be seen that the participants did not seem to notice that it was impossible to give 1 as a reasonable answer.

**Subtraction with a whole number:**  $1 - \frac{2}{6}$ . Unlike the first subtraction task, only 27 participants (less than 50%) gave the correct answer for this task. However, the most common error (E1) occurred when 29 participants left their answer as  $\frac{4}{6}$  without simplifying it. Thus, most of the participants were able to work through the subtraction procedure, but they did not present the answer in such a form, which was defined as correct in this study. One student teacher simplified the fraction first from  $\frac{2}{6}$  to  $\frac{1}{3}$ , but after subtracting  $1 - \frac{1}{3}$ , he/she gave the answer in decimal form (0.666). Two participants used colored circles, and one of them arrived at the correct answer. One participant left the task blank.

Mathematical writing errors E2 were also common with this task; ten participants used mathematical symbol writing incorrectly, and some had missing steps in their solutions. Moreover, three participants used a procedurally correct but an

unnecessarily long solution method (E5):  $1 - \frac{2}{6} = \frac{1}{1} - \frac{2}{6} = \frac{1 \cdot 6}{1 \cdot 6} - \frac{2 \cdot 1}{6 \cdot 1} = \frac{6}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$ .

After converting the whole number 1 to a fraction form, they multiplied both fractions to get 6 as a common denominator, even though there was no need to multiply the latter fraction by 1. This seemed inefficient, and the participants seemed to do this routinely without thinking about the meaning of multiplying by 1.

**Multiplication with different denominators:**  $\frac{3}{4} \cdot \frac{2}{5}$ . Only 22 student teachers (37%) gave the correct answer by showing some mathematical steps in this task. Three participants used decimals, but they arrived at three different incorrect answers. Ten students left this task blank, which may indicate that they were more uncertain with multiplication than with the operations in the previous tasks.

The difficulty with the multiplication operation was seen also with the number of participants making procedural E6 errors. Eleven participants cross-multiplied the numerators and denominators, which they did in two different ways:  $\frac{3}{4} \cdot \frac{2}{5} = \frac{4 \cdot 2}{3 \cdot 5} = \frac{8}{15}$  or  $\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 5}{4 \cdot 2} = \frac{15}{8} = 1 \frac{7}{8}$ . Interestingly, one participant used a correct multiplication algorithm first but then crossed it out and used the latter of the faulty methods presented in previous the example.

Another E6 error in the multiplication operation was the use of common denominators, even though this was unnecessary. Altogether, seven participants multiplied both fractions to get 20 as the common denominator. One of them gave  $\frac{120}{400}$  as an answer; the others kept 20 as the denominator after multiplying the numerators and arrived at a procedure as follows:  $\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 5}{4 \cdot 5} \cdot \frac{2 \cdot 4}{5 \cdot 4} = \frac{15}{20} \cdot \frac{8}{20} = \frac{120}{20} = \frac{60}{10} = 6$ . Again, the participants seemed to be uncertain about the role and use of denominators, and they did not notice that a whole number solution was an impossible answer for this task.

Interestingly, none of the participants who correctly solved the multiplication  $\frac{3}{4} \cdot \frac{2}{5}$  used the option of simplifying the numbers 2 and 4 before multiplying across the numerators and denominators. This can be interpreted as a rote understanding of the algorithm or a limited number sense when seeing multiple numbers.

**Division by a whole number:**  $\frac{3}{4} / 3$ . Similar to the results in multiplication, 22 participants gave the correct answer for the division task. Six of them used the mathematical invert-and-multiply procedure and showed the steps that led to the correct solution. Two participants first converted the divisor 3 to fraction form and then wrote the correct answer. However, it was not possible to find out whether they followed the correct division procedure or whether they just divided across since they wrote as follows:  $\frac{3}{4} / 3 = \frac{3}{4} / \frac{3}{1} = \frac{1}{4}$ . Moreover, five participants used decimals in their

solutions; two of them arrived at the correct answer in fraction form and one gave a right answer as decimals. Like in the previous tasks, the participants using pictures were more successful in finding the correct answer than those who used mathematical symbol representations but made errors in them. However, it was difficult to find out the mathematical thinking model behind the correct answer in these pictorial representations as well. For example, it is unclear whether the answer in [Figure 7](#) refers to one of the colored parts in the rectangle or to the remaining white part.

$$\frac{3}{4} / 3 = \boxed{\text{shaded part}} / 3 = \frac{1}{4}$$

Figure 7. A pictorial solution for the division task (participant 11)

In general, solving the fraction division task by showing their solution steps seemed challenging for the student teachers. A total of eighteen participants made mathematical writing errors (E2), and similar to the multiplication task, ten participants left the division task blank; four of them did this in the case of multiplication as well. In addition to these technical errors, even six different error subtypes that were made altogether by twenty participants were found for the division operation. The most common of these procedural E7 errors occurred when the whole number divisor 3 was converted to fraction form. Some participants seemed to prefer having the same denominators for both the dividend and divisor even though it was unnecessary, and thus, eight of them converted the divisor to  $\frac{12}{4}$  and one incorrectly to  $\frac{4}{4}$ ; four participants also changed the divisor 3 to the form  $\frac{3}{3}$ . Interestingly, only two of those who used the form  $\frac{12}{4}$  went further in their solutions but they arrived at the different incorrect answers presented in [Figure 8](#).

$$\frac{3}{4} / 3 = \frac{3}{4} / \frac{12}{4} = \frac{4}{4} = 1 \quad \frac{3}{4} / 3 = \frac{3}{4} / \frac{12}{4} = \frac{3}{4} \cdot \frac{4}{12} = \frac{12}{48} = 3$$

Figure 8. Incorrect solutions for division (participants 15 and 49)

As can be seen in the examples above, the participants made multiple errors in their solutions; in the example on the left, the student teacher has obviously divided

the numerator 3 by 12 and kept the denominators to get  $\frac{4}{4}$ , whereas the other student teacher seems to use the invert-and-multiply procedure, but then incorrectly divides 48 by 12. Other procedural errors for the division operation were (a) dividing the numerator or both the numerator and denominator by the whole number divisor, (b) first multiplying the numerator and denominator by the divisor and then dividing the new fraction by it, (c) dividing across by a fraction form divisor, and (d) cross-multiplying by the inverted divisor. Similar to addition with different denominators, the number of different incorrect solution methods in the division task seems to indicate that the participants are guessing the solution methods when they do not remember or understand the correct algorithm; some participants even wrote on the research questionnaire that they did not remember how to divide fractions.

In this section, the participating student teachers' solutions for fraction tasks were described in general and in terms of their errors and difficulties with the six routine fractions tasks. The analysis revealed several limitations in their CCK on fractions and also some other limitations in their basic knowledge of mathematics; these findings were not directly connected to their knowledge of fractions. In the next section, the most important results of this study will be summarized and discussed.

## 6 Discussion and conclusions

In this study, student teachers' CCK on fractions was investigated by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. Many of the findings concerning their procedural errors in fraction operations are in line with findings in previous studies (e.g., Newton, 2008; Van Steenbrugge et al., 2014; Young & Zientek, 2011). In other words, the participants in this study had difficulties with all fraction operations and especially with division and multiplication. Many of them seemed to have a rule-based and rote understanding of the algorithms, and they used several incorrect methods for their solutions. Moreover, they seemed to lack knowledge of using other representations when not being able to use a correct algorithm. It was also seen in this study that student teachers have difficulties in using fraction number sense.

Different problems concerning the teaching and learning of fractions have been reported for decades, and the need to develop student teachers' knowledge of fractions has also been reported earlier (e.g., Van Steenbrugge et al., 2014). This study is

consistent with the previous findings about student teachers' limited CCK of fractions. In addition, the study reveals some other limitations in their mathematical CCK.

In general, it was surprising that so many of the participating student teachers made several types of errors and that there was so wide difference between the participants when solving the fraction tasks. The participants were expected to be familiar with the routine tasks and the fraction content included in the tasks, since they had recalled and repeated this content in their previous mathematics course in teacher education. The uncertainty that many participants demonstrated in their CCK was seen in the number of tasks left blank and, for example, in their lack of using different fraction forms coherently throughout the solutions. Moreover, showing how to solve a routine task step-by-step seemed to be challenging for most of the student teachers; the more steps needed to find a solution, the more difficult it became to write out the procedures and the more errors the participants made. Like student teachers in Jakobsen et al.'s study (2014), many participants used in their solutions incorrect mathematical notations and moreover, they used separate solution steps that formed illogical statements without constructing a logical solution procedure.

The participants in this study also demonstrated limitations in their basic knowledge concerning mathematical symbol writing and the use of different representation forms. This is an important finding since these errors did not seem to be directly connected to fractions but rather they seemed to be general limitations in student teachers' CCK, which may have an effect when student teachers work with fraction as well. For example, some of the student teachers were misusing the equal sign, and they made errors in differentiating the symbols to simplify a fraction and to divide it. Making this kind of errors in their mathematics teaching might be confusing for elementary school students. Unlike Newton's study (2008), where none of the 85 participants used pictures to solve routine fraction tasks, seven participants in the present study used pictures to find the correct answers. However, it seemed that pictorial representations were used with tasks where the participants were uncertain about the correct algorithm, and many of the pictures that they presented could be seen as they mental images of the fractions and not as representations of the solution procedures needed for the tasks. As Moss et al. (1999) have stated, especially the use of pie charts may be misleading in elementary mathematics teaching. Thus, it seems that the becoming teachers need to learn how to better use pictorial representations to visualize abstract mathematical procedures. Moreover, a robust knowledge of correct mathematical algorithms is needed as well since pictorial illustrations with

simple fractions such as  $\frac{3}{4}$  and  $\frac{1}{2}$  work well, but the use of pictures becomes complicated for fractions like  $\frac{13}{41}$  and  $\frac{11}{21}$ . Some participants in this study used also decimals throughout the fraction tasks but they did not seem to notice the errors that occurred in their solutions when they converted improper fractions to decimals (c.f. Muir & Livy, 2012).

Moreover, many student teachers in this study did not notice their incorrect statements and unreasonable answers even in the simplest cases. However, determining equivalence and judging the reasonability of answers are essential parts of fraction number sense (Lamon, 2020) and CCK for mathematics teachers in their daily work (Ball et al., 2008). This finding like the previous one concerning mathematical symbol writing and using different representation forms may not be connected to fraction tasks only and should therefore be researched further.

Further, an interesting finding was that the participating student teachers seemed to guess at which algorithm to use when they did not remember or understand the correct solution method. Often, they seemed to remember some separate steps of the algorithms instead of understanding the procedures as a whole. Also, as Newton (2008) states, it seems that even though student teachers remember many procedures, they use them in inappropriate ways with fractions. For a mathematics teacher, a robust CCK goes beyond rote learning and memorization of algorithms since “teaching requires knowledge beyond that being taught to students” (Ball et al., 2008, p. 400).

Although student teachers do not need to hold a level of expertise equivalent to that of an experienced elementary mathematics teachers, they should not be regarded as novices in their mathematical CCK. However, student teachers may enter their studies in teacher education with different prior mathematical knowledge and with different kinds of experiences in mathematics teaching and learning. As seen in this study and in previous research (e.g. Newton, 2008), not all student teachers are competent in their basic knowledge of fractions, and the limitations found in their CCK may not predict success in teaching of fractions in their future profession as elementary mathematics teachers (Van Steenbrugge et al., 2014). Thus, teacher educators need to pay attention to student teachers’ individual differences and to be aware of their different error patterns (Young & Zientek, 2011). Especially, the results in this study reveal that student teachers need a deep knowledge of fractions and mathematical symbol writing and the meaning of the procedures as well; it is not enough to be able to produce correct answers for mathematical tasks. To enhance this



knowledge and student teachers' ability to interpret others' mathematical solutions as well student teachers should be given fraction tasks to be solved in different ways like Jakobsen et al. (2014) and Maciejewski and Star (2016) conclude in their studies.

The present study, conducted in the Swedish context, confirms the results from other countries during recent decades. Thus, it can be stated that there is still much to do when developing student teachers' CCK on fractions and other mathematical content as well. Since the present study concerned only a group of student teachers in one Swedish university, a limitation of the study is the inability to generalize the results beyond this population. However, some errors did occur across the participants, and this may rise questions about general difficulties in student teachers' CCK. For example, student teachers' use of mathematical symbol writing and mathematical representations for topics other than fractions could be addressed in further research. Moreover, maybe the biggest challenge in teacher education is how to address student teachers' individual differences and their various difficulties in mathematics.

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