How to develop a structural conception of algebra of school students

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Abstract: Algebra plays a pivotal role in school students' mathematics learning. Students, however, have serious difficulties especially with 'seeing' the algebraic structures and expressing generality with algebraic symbols. These skills are not gained accidently; developing them in students takes continuous, sustained, and focused effort from both students themselves and teachers. As teachers of mathematics, what kind of classroom experience could we provide our students to prepare them for using algebra successfully? How could we assist them to develop structural conceptions of algebra? In this paper, I describe an approach based on the Mathematical Habits of Mind framework (Cuoco et al., 1996) as a response to these questions and provide examples of how this approach may look in the classroom.

Keywords: algebra, generalising and justification activities, professional learning, secondary teachers

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1 Introduction

The mathematics classroom is the primary context in which students gain algebra knowledge and skills. Given a classroom environment in which they are supported to 'see' the structures in algebra, I believe every student can learn algebra. Whilst researchers have found that students' algebra knowledge and skills can be improved with appropriate teaching approaches (Star et al., 2015), the teaching practices in many mathematics classrooms do not support students in learning algebra. Both national and international studies have shown that the majority of Australian school students perform poorly in algebra, like many students in several other countries I assume. Thomson et al. (2020) report that Year 8 Australian students' performance in TIMSS 2019 was the weakest in Algebra content domain among other content domains such as Number and Data. Sullivan (2011) states a majority of Year 9 Victorian students were unsuccessful in answering the question: $2\times(2x-3) + 2 + ? = 7x-4$. This article broadly concerns the possibility of making algebra accessible to school students; it specifically concentrates on how students can be supported to develop more structural conceptions of algebra. Here, influenced by Star et al. (2015) and Couco (2012), structure refers to the algebraic pattern or rule underlying a numerical relation, while a structural conception of algebra refers to seeing the patterns in repeated calculations and being able to use algebraic language to show generality.

2 The context for the paper

Creating opportunities for students to gain algebra knowledge and skills requires teachers to hold these skills and knowledge themselves. Since 2019, as a group of researchers in Tasmania (Australia), we have aimed to contribute to enhancing teacher educators' (Hatisaru et al., 2020) and secondary school teachers' proficiency with algebra teaching (e.g., Hatisaru et al., 2022). As continuation of these works, in 2021, I established a teacher study group in the same State which two participant secondary school teachers solved and discussed algebra problems. The aim was to develop a deeper understanding of algebraic processes and solution strategies and examine the effectiveness of teacher study groups as a teacher professional learning approach. Study group meetings were held virtually every three to four weeks throughout an academic year, and each meeting lasted for an hour. From September 2021 to March 2022, we had seven meetings. One week before each meeting, I sent group members two or three algebra problems for them to solve first and anticipate how students would solve them, or how they would teach the problems in their classroom. Anticipated student work, or the ways of teaching the problems, guided the substance and direction of discussions.

Kilpatrick et al. (2001) classify the activities in school algebra into three: representational activities (e.g., word problems), transformational or rule-based activities (e.g., solving equations), and generalising and justifying activities (e.g., noting structure, proving). This classification by Kilpatrick et al. (2001) is echoed in Kieran's (2007) GTG model, where the activities of school algebra are grouped into three aspects: generational, transformational, and global/meta-level. The problems posed in the teacher study group fit the algebraic activities of representing, transforming, and generalising.

As I engaged with teachers on the algebraic activities and how to teach them, as a mathematics teacher educator, I refined my understanding of algebraic activities, analysed the teachers' solutions to the problems, and considered which teaching approaches would best support student learning in algebra. It was my observation that generalisation and justification – or global/meta-level – activities (the focus of this article) were the most challenging algebraic activities to the teachers, both to deal with and to teach.

2.1 Generalising and justification activities

In the last two meetings, the study group engaged in generalising and justification activities. One of the scholarly works on the focus of these activities was the article Teaching and Learning of School Algebra by Carolyn Kieran published in 1992. One week before the final meeting I sent the following three problems to the group, as all cited in Kieran (1992, p. 407):

> Problem #1: Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it? (Chevallard & Conne, 1984)
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> Problem #2: A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Using algebra, show that she is right. (Lee & Wheeler, 1987)
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> Problem #3: Show, using algebra, that the sum of two consecutive numbers is always an odd number. (Lee & Wheeler, 1987)

Problem #1 was used by Chevallard and Conne (1984) in their research investigating students' capability to use letters to express the general, reported in Kieran (1992). As a sequence of problems, the researchers posed this problem to an eighth-grade student and found that the student had a well-developed structural conception of algebra (he was the only student in the study holding a structural conception of algebra). The student first worked with three consecutive numbers: 3, 4, 5 and then with 10, 11, 12. In both cases he found the same result: 1. When asked what would happen if algebra was used, the student replaced all the given numbers by unknowns and wrote: $x^2 - y.z = 1$ but then realised that using only one unknown would be better and came up with the algebraic equation $x^2 - [(x + 1).(x - 1)] = 1$ governing the correct numerical relation in the problem. The two study group teachers solved the problem in the same as this student: both of them worked with numbers first and then generalised the rule by using an unknown (x or n).

Lee and Wheeler (1987) used Problems #2 and #3 in their study exploring students' conceptions of generalisation and justification, stated in Kieran (1992). The slight difference between these three problems is that Problems #1 and #2 require study with numbers initially while Problem #3 does not necessarily require that. Lee and Wheeler (1987) surveyed 354 tenth-grade students and conducted follow-up interviews with 25 of the students. They found that the students valued the use of algebra to express the general, but a great majority of them had a procedural conception of algebra. That is, they predominantly worked with numbers to show the general, but they were less successful in demonstrating the rule by using algebraic language. This was not the case for the two study group teachers, of course. Both of them were successful in the use of algebraic language in solving these problems. In solving Problem #3, for example, one of the teachers assigned an unknown to two consecutive numbers (x and $x+1$) and then checked the general rule ($2x + 1$) with the numbers (5 and 6). The other teacher did not even need to work with numbers and directly expressed the numerical relations in the problem by an algebraic rule: assuming that the numbers are n and $n + 1$, their sum $2n + 1$ always gives us an odd number.

2.2 Problem situation

Based on the poor student responses in previous research studies, including the two studies mentioned in this section, Kieran (1992) raised a concern saying that:

> The challenge for algebra instructors is to find a means of making the structural aspects of algebra accessible to a greater percentage of students (p. 408).

As both of the teachers seemed to be proficient with the use of algebraic language to respond to these two problems, I shifted my focus from the teachers' proficiency with solving the two problems to their proficiency with the teaching of them. In the meeting, I raised Kieran's concern that had been reflected three decades ago: How could we assist students in developing structural conceptions of algebra? How, for instance, could these problems be taught in the classroom to contribute to that goal? We paused the meeting and had 10 to 15 minutes time to individually develop some ideas. Eventually, one of the teachers said that he could use some web-based resources or concrete materials such as Lego blocks or counters in the teaching of Problem #3, and the other teacher said that she could not think of any strategy.

Using concrete materials (e.g., counters, blocks) or visual models could be useful to solve Problem #3. In their book entitled Five Practices for Orchestrating Productive

Mathematics Discussion, Smith and Stein (2011) give a similar problem to Problem #3: "Explain why the sum of any two odd numbers is always even" (p. 46) and present three possible solutions to the problem: concrete model, logical argument, and algebraic proof. In concrete model solution, a pictorial model is used. That is, 5 and 11 counters are organised to show, for any set of two odd numbers (e.g., 4 counter + 1 counter, and 10 counter + 1 counter), the extra counters will always make a pair (1 counter + 1 counter). Although concrete solution strategies or "representational-based arguments" are powerful and can engage young students in the process of proof (Schifter, 2009, p. 84), they do not describe the structural aspects of algebra enough. Students should learn more abstract or symbolic arguments, in addition to any type of concrete experiences.

During the discussions in the meeting and later I considered the issue. As teachers of mathematics, what kind of classroom experiences could we provide our students to prepare them for using algebra successfully to represent the general rules in numerical relations? How could we assist them to develop structural conceptions of algebra? Below, I describe an approach based on the Mathematical Habits of Mind (Couco et al., 1996) framework as a response to this self-questioning.

3 The Mathematical Habits of Mind framework

I found the Mathematical Habits of Mind (MHoM) framework useful for interpreting my approaches to teaching algebra. This well-regarded framework in the field of mathematics education was created by Couco et al. (1996) to inform mathematics teaching practices and curricula and course construction efforts. The MHoM approach shifts the common descriptions to the question of what mathematics is about from: 'it is about geometry, doing arithmetic, or solving equations' to: 'it is about seeing patterns, connecting concepts, or ways for solving problems'. Here, it is aimed to develop mental habits in students that can provide them with a repertoire of general strategies (heuristics) and ways that can be applied in a variety of situations (Couco et al., 1996). The MHoM include guessing, challenging (even correct) solutions, looking for patterns, using alternative representations, classifying, and thinking algebraically (Levasseur & Cuoco, 2003). These mental habits can support the mathematical approaches; it is desirable if all high school graduates have them (Couco et al., 1996).

Algebra is a language for expressing mathematical ideas, and it consists of more than a tool to represent mathematical objects with symbols (Couco et al., 1996). Along with general MHoM, Cuoco and colleagues identified thinking habits specific to algebra that are centre around representing, transforming, and generalising the symbols. This means students who have algebraic capabilities use a special collection of mental habits such as performing calculations, using abstraction, using algorithms, breaking things into parts, extending things, and representing things. According to Cuoco (2013), however, students have most the difficulty with *seeking regularity in repeated calculations* and *expressing that generality with algebraic symbols* (Cuoco, 2012). A different but related note here is these sometimes make little sense to even teachers (Cuoco, 2013). In this paper, I focus on these two mathematical practices that are at the core of the practice of algebra.

3.2 A structural-based approach to Problems # 1 to #3

Consider Tables 1 to 3 given to present the solutions to Problems #1 to #3. All three tables are organised to assist students to seek the numerical relations in the problems and translate those relations in symbolic language by representing them with symbols. The tables illustrate what this process looks like. It is important to note that, attention is given to represent the numbers according to the underlying mathematical feature, in order to make the relevant numerical relation flows into an algebraic demonstration.

In Problem #1, if the number is 3, the next consecutive number is represented 3+1 rather than 4, and the next consecutive number is represented 3+1+1 rather than 5 (see Table 1). It is what makes the approach a structural-based approach.

Table 1. A structural-based approach to Problem #1

Problem #1: Take three consecutive numbers. Calculate the square of the middle one, subtract from it the product of the other two. Do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it? (Chevallard & Conne, 1984; cited in Kieran, 1992)

Solution:

The number	The next consecutive number	The next consecutive number	The square of the middle number	The product of the other two numbers
3	3+1	3+1+1	$(3+1)^2 = 3^2+3\times2+1$	$3\times(3+1+1) = 3^2+3\times2$
5	5+1	5+1+1	$(5+1)^2 = 5^2+5\times2+1$	$5\times(5+1+1) = 5^2+5\times2$
8	8+1	8+1+1	$(8+1)^2 = 8^2+8\times2+1$	$8\times(8+1+1) = 8^2+8\times2$
9	9+1	9+1+1	$(9+1)^2 = 9^2+9\times2+1$	$9\times(9+1+1) = 9^2+9\times2$
…	…	…	…	…
n	n+1	n+1+1	$(n+1)^2 = n^2+n\times2+1$	$n\times(n+1+1) = n^2+n\times2$

The claims depicted in Table 1 are:

- if the first number is 3, the next consecutive number is 3+1, the next consecutive number is 3+1+1;
- the square of the middle number is $(3+1)^2$ which can be represented as $3^2+3\times2+1$;
- the product of the other two numbers is $3\times(3+1+1)$ which can be represented as $3^2+3\times2$;
- the difference between the square of the middle number and the product of the other two numbers is $[3^2+3\times2+1] - [3^2+3\times2] = 1$.

The table helps these numerical relations naturally flow into a general algebraic formula or rule:

- if the first number is n, the next consecutive number is $n+1$, and the next consecutive number is $n+1+1$;
- the square of the middle number is $(n+1)^2$ which can be represented as $n^2+n\times2+1$;
- the product of the other two numbers is $n\times(n+1+1)$ which can be represented as $n^2+n\times2$;
- the difference between the square of the middle number and the product of the other two numbers is $[n^2+n\times2+1] - [n^2+n\times2] = 1$.

Problem #2 gives a good deal of practice in representing things in their equivalent forms. Rather than doing automatic calculations (e.g., if the original number is 2, $2 \times 5 = 10$; $10 + 12 = 22$; $22 - 2 = 20$; $20 \div 4 = 5$; and 5 is 3 more than the original number), it is important that we make use of structures and perform calculations such a way that they could lead to a symbolic expression holding the generality in this situation. Trying some numbers in that way like -3, -2, 0, 3, 5, and 7 leads to that algebraic expression. As depicted in Table 2:

- if the first number is -3 (equals to -3×1), multiplying it by 5 gives (-3×5); adding 12 to this number gives $(-3\times5)+12$; subtracting the original number gives $(-3\times5)+12 - (-3\times1)$; finally, dividing this number by 4 gives $\frac{(-3\times5)+12-(-3\times1)}{4}$;
- $\frac{(-3\times5)+12-(-3\times1)}{4}$ is equal to $\frac{(4\times-3)+12}{4}$ because 1 lots of -3 is taken away from 5 lots of -3 which gives 4 lots of -3 (i.e. 4×-3);

- the equivalent form of $\frac{(4\times-3)+12}{4}$ is $\frac{(4\times-3)}{4}+\frac{12}{4}$ as the vinculum (horizontal fraction bar) acts for both the numerical expression in the numerator and number in the denominator;
- that gives the answer: -3+3, which is 3 more than the original number.

Trying the other numbers and keeping track of these steps as depicted in Table 2 help to see the regular steps in calculations and lead to expressing the pattern in a general form:

- if the first number is n (equals to $n\times1$), multiplying it by 5 gives ($n\times5$); adding 12 to this number gives ($n\times5$)+12; subtracting the original number gives ($n\times5$)+12 – ($n\times1$); finally, dividing this number by 4 gives $\frac{(n\times5)+12-(n\times1)}{4}$;
- $\frac{(n\times5)+12-(n\times1)}{4}$ is equal to $\frac{(4\times n)+12}{4}$ because 1 lots of n is taken away from 5 lots of n which gives 4 lots of n (i.e. $4\times n$);
- the equivalent form of $\frac{(4\times n)+12}{4}$ is $\frac{(4\times n)}{4}+\frac{12}{4}$ and that gives $n+3$; that is, 3 more than the original number.

Table 2. A structural-based approach to Problem #2

Problem #2: A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, “I think that would happen, whatever number I started with.” Using algebra, show that she is right (Lee & Wheeler, 1987; cited in Kieran, 1992).

Solution:

A number	Multiply by 5, add 12, and subtract the original number	Its equivalent form	Its equivalent form	The answer
-3	$\frac{(-3\times5)+12-(-3\times1)}{4}$	$\frac{(4\times-3)+12}{4}$	$\frac{(4\times-3)}{4}+\frac{12}{4}$	-3+3
-2	$\frac{(-2\times5)+12-(-2\times1)}{4}$	$\frac{(4\times-2)+12}{4}$	$\frac{(4\times-2)}{4}+\frac{12}{4}$	-2+3
0	$\frac{(0\times5)+12-(0\times1)}{4}$	$\frac{(4\times0)+12}{4}$	$\frac{(4\times0)}{4}+\frac{12}{4}$	0+3
3	$\frac{(3\times5)+12-(3\times1)}{4}$	$\frac{(4\times3)+12}{4}$	$\frac{(4\times3)}{4}+\frac{12}{4}$	3+3
5	$\frac{(5\times5)+12-(5\times1)}{4}$	$\frac{(4\times5)+12}{4}$	$\frac{(4\times5)}{4}+\frac{12}{4}$	5+3
7	$\frac{(7\times5)+12-(7\times1)}{4}$	$\frac{(4\times7)+12}{4}$	$\frac{(4\times7)}{4}+\frac{12}{4}$	7+3
…	…	…		…
n	$\frac{(n\times5)+12-(n\times1)}{4}$	$\frac{(4\times n)+12}{4}$	$\frac{(4\times n)}{4}+\frac{12}{4}$	$n+3$

The same approach is applied to solve Problem #3. Table 3 depicts that (for example) if the number is 1, the next consecutive number is 1+1, and the sum of the two numbers is 1+1+1. If the number is 2, the next consecutive number is 2+1, and their sum is 2+2+1. As in Table 1, this numerical relation, represented in a form that make 1+2 = 1+1+1 = 2×1+1 visible, can flow into an algebraic demonstration: if the number is n, the next consecutive number is n+1, and their sum is n+n+1 = 2×n+1 which governs odd numbers.

Table 3. A structural-based approach to Problem #3

Problem #3: Show, using algebra, that the sum of two consecutive numbers is always an odd number. (Lee & Wheeler, 1987; cited in Kieran, 1992)

Solution:

The number	The next consecutive number	The sum of the numbers	
1	1+1	1+1+1	2×1+1
2	2+1	2+2+1	2×2+1
3	3+1	3+3+1	2×3+1
4	4+1	4+4+1	2×4+1
5	5+1	5+5+1	2×5+1
…	…	…	…
n	n+1	n+n+1	2×n+1

4 Concluding words

Students need to be supported to gain useful ways of thinking about mathematical content. When mathematics teaching is planned around translating numerical relations in symbolic representations, a structural conception of algebra could be developed in students. When planning the lessons, teachers may ask: Am I sure this could make the structural aspect of algebra visible? How that aspect can be made available? In this way, teachers would be challenging themselves to consider how to advance students' numerical skills through algebraic skills.

In most classes, students have opportunities to 'learn' algebra (basically practicing procedures), but they do not often have opportunities to 'see' algebraic structures. I believe that the structural-based arguments captured in Tables 1 to 3 can provide a mechanism that is accessible and powerful for students to engage in the process of generalisation and using the language of algebra. Alongside the goal of helping the study group teachers teach generalisation and justifying algebraic activities through structural-based approaches, Tables 1 to 3 were presented to them and discussed. It is interesting to mention here that the teachers remarked that they learnt much from the approach taken in the tables.

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