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Acceptability of an argument in the mathematical classroom: the role of beliefs

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Abstract: The study focuses on upper secondary students' beliefs which emerge in students' evaluating of solutions strategies and of the connected arguments proposed by their peers. In particular, an activity aimed to introduce the concept of classical probability and based on the Problem of Points is considered. The thematic analysis is conducted on students' protocols and on transcriptions of whole class discussions. Five themes of beliefs are defined. The analysis shows a great variety of beliefs which are crucial to evaluate others' solutions and arguments, but not sufficient to reject incorrect solutions. The themes of beliefs are confronted with Goldin's categories to highlight their variety.

Keywords: students' beliefs, argumentation, probability, classroom discussion, upper secondary school

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1 Introduction and theoretical framework

Argumentation is a crucial aspect in Mathematics Education and, more broadly, in everyday lives. This concept emerges strongly in the Ancient Greek tradition and evolves through history (Dutilh-Novaes, 2021). For a significant period, under the influence of scholars like Descartes, argumentation was considered valid only when tied to specific deductive and inferential steps. This perspective led scholars to primarily focus on purely cognitive aspects, overlooking those associated with the affective sphere. Changes to this perspective emerged in the second half of the last century, particularly with the works of Toulmin (2003) and Perelman and Olbrechts-Tyteca (1989). Both works emphasized that deductive systems are not sufficient to describe every sound argument human beings may produce. Perelman and Olbrechts-Tyteca (1989) delved into the role of interlocutors and highlighted how it is crucial to consider not only cognitive aspects but also affective ones, such as interlocutors' beliefs and values, in developing arguments that are effective for a particular audience. These works have influenced the field of Mathematics Education, especially in developing an idea of argumentation as a *process* in which students and teachers exchange and develop arguments (e.g. Krummheuer, 1995; Yackel and Cobb, 1996). However, at my knowledge, despite the numerous research about argumentation in Mathematics Education, the role played by affective factors in accepting or refusing arguments remains under-investigated.

To frame this interactive argumentative process, I rely on the Mercier and Sperber's studies (2017) developed in the field of cognitive science. Accordingly, interlocutors' beliefs strongly influence the argumentative processes. For this reason, I chose to explore the role of students' beliefs in a mathematical classroom argumentative process aimed at reaching consensus about a mathematical problem solution.

The role of beliefs has been extensively studied in Mathematics Education. Concerning problem solving and reasoning, Schoenfeld (1983) has highlighted that beliefs play a crucial role when individuals solve mathematical problems. Beliefs could hinder the problem-solving process undertaken by students (e.g., Schoenfeld, 1992), or they can sustain students to succeed, for example, helping them to persist in finding a solution that was not immediate for them (Carlson, 1999). Affect-related factors, such as beliefs, and cognitive factors are intertwined during problem-solving (Furinghetti & Morselli, 2009). Finally, Sumpter (2013) delves into the role of beliefs in reasoning, which is considered to be a "line of thought adopted to produce assertions and reach conclusions in task solving" (p. 1120). Her investigation shows three main themes of beliefs (safety, expectations, and motivation) that influence students' reasoning. Reasoning is tightly connected to argumentation, although some generally accepted distinction. In fact, reasoning is generally considered to belong to the realm of thinking, while argumentation falls within the domain of language and communication. Nevertheless, their strong connection is undeniable. This explorative study aims to broaden the investigation of beliefs in Mathematics Education considering the realm of argumentative processes. To sum up, despite argumentation and beliefs being deeply addressed in Mathematics Education, at my knowledge, their interconnection is not equally explored. In this paper I then present an explorative study about the role of beliefs (which are at centre of the investigation and of the analysis) in argumentative processes in mathematical classroom (which frame the research).

1.1 Beliefs

The term beliefs has not an universally accepted definition. In this explorative study I chosen to follow the broad definition by Goldin (2002) considering "beliefs as multiply-encoded cognitive/affective configurations, usually including (but not limited to) prepositional encoding, to which the holder attributes some kind of truth value" (Goldin, 2002, p. 64). Where "truth" is not limited to a symbolic-logic-related sense, but is considered in a wider sense, including for instance religious truth, conventions or applicability. Beliefs are generally not isolated but interconnected, forming individual structures. These are labelled as systems of beliefs if they are socially shared. Goldin propose eleven categories in which structures or systems of beliefs can be distinguished by their content. Those are:

> [1] Beliefs about the physical world, and about the correspondence of mathematics to the physical world (e.g., number, measurement); [2] Specific beliefs, including misconceptions, about mathematical facts, rules, equations, theorems, etc. (e.g., the law of exponents, the quadratic formula, the idea that "multiplication always makes larger"); [3] Beliefs about mathematical

> validity, or how mathematical truths are established; [4] Beliefs about effective mathematical reasoning methods and strategies or heuristics; [5] Beliefs about the nature of mathematics, including the foundations, metaphysics, or philosophy of mathematics; [6] Beliefs about mathematics as a social phenomenon; [7] Beliefs about aesthetics, beauty, meaningfulness, or power in mathematics; [8] Beliefs about individual people who do mathematics, or famous mathematicians, their traits and characteristics; [9] Beliefs about mathematical ability, how it manifests itself or can be assessed; [10] Beliefs about the learning of mathematics, the teaching of mathematics, and the psychology of doing mathematics; [11] Beliefs about oneself in relation to mathematics, including one's ability, emotions, history, integrity, motivations, self-concept, stature in the eyes of others, etc. (Goldin, 2002, pp. 67–68)

Although these categories are not detailed, they offer an insight about the different beliefs related with mathematics that could be considered. In this exploration, I chosen to take into consideration these categorise to investigate whether some of them can be recognised in the analysis of beliefs that emerge in the classroom discussion, helping me distinguish between the different beliefs that emerge. The categories are, in my opinion, not necessarily exclusive and perhaps others could be considered as well.

1.2 Argumentation and the role of beliefs in it

Argumentation is here conceived as a social process, co-developed by students and teachers in their classrooms. In the context of the study, this collaborative process aims at reaching consensus about a problem solution. What is accepted as a final solution strictly depends on this dialectic process.

Argumentations are contextual. In classroom several factors can play a role in the developing an argumentative process, such as the function of argumentation in classroom – which can be considered explanatory (Ferrari & Saccoletto, 2023), the educational aims (Saccoletto, 2023), and – naturally – the interlocutors, with their knowledge and their beliefs. In here the focus is on the role of beliefs.

To frame all these aspects, Mercier and Sperber’s (2017) perspective is followed. According to the two scholars, in argumentations, humans invoke reasons for social motivations, such as *explaining* or *justifying* themselves to others. According to the authors, some of the reasons people appeal to are good enough to understand why someone thought or decided something, or why they acted in a certain way, but they are not enough to share their line of thinking, acting, or deciding. This is because reasons are psychological reasons, i.e. they are mental representations of facts. Facts himself do not have casual power, while psychological reasons do. The same fact can be a good reason to different conclusion. For example, if it is snowing, we can conclude that it is better staying home, or, on the contrary, go out, maybe skiing. This changes from individual to individual. If we want to justify our standpoints to others and fulfil a *justificatory function*, we should consider facts that are acknowledged as good reasons by the interlocutors. This depends by interlocutors’ knowledge and beliefs. In fact, to accept or refuse an argument, our beliefs could play a role. Hence, to be effective, the speakers should connect her claim with interlocutors' beliefs and knowledge. The interlocutors should be able to follow and evaluate the connection.

This perspective allows to address the role of interlocutors' beliefs in accepting or refusing a mathematical argument.

1.3 Research questions

The research questions are the following: Which beliefs are used by upper secondary students to support or oppose mathematical arguments that are expressed during the solution of a specific problem (the Problem of the Points)? How can these beliefs be positioned in the categories outlined by Goldin?

2 Method

2.1 Design of the study

In this study an activity implemented in an upper secondary school classroom of 16 students (Grade 10th) is considered. Students are introduced to probability thinking thorough the Problem of Points, traditionally linked to the genesis of the classical theory of probability (Todhunter, 2005). The problem addressed by students is:

> Two players A and B play heads or tails with a fair coin. Each game, corresponding to each coin toss, is won by A if the outcome of the toss is heads and by B if the outcome is tails. A and B give 12 euros each. The stake is 24 euros. The player who first wins 6 rounds wins the game, and thus the entire stake. A always bets on "heads" and B on "tails". The game is interrupted at the score 1- 0 for A. How should the stakes be fairly divided i.e., that it gets both players to agree?

For the entire activity, students work in four groups of four students each. Students are asked to i) solve the problem in small groups; ii) presenting and justifying their solution to others; iii) return in small groups to reflect on other groups' solutions; iv) sharing their reflection with the whole class and discussing acceptable solutions and justifications. In the third step of the activity students reflect on their and others' solutions. Particularly, they are asked to list the greatest strengths and weaknesses of each of the solutions proposed by the other groups, and to discuss whether others' arguments should be accepted or refused.

2.2 Analysis

The four steps of the activity have been video-recorded and groups' protocols have been collected. For the analysis I focused on video transcripts of peers' interventions and remarks right after the presentation of each group (step i), on groups' protocols produced in step three, and on video transcripts of the last discussion (step iv), in which students shared their reflections about others' arguments. The focus of the analysis was the reasons chosen by students to criticize or agree with others' solutions, I considered those reasons linked with students' beliefs, as explained trough the example. The reasons for accepting or refusing an argument that were proposed by the teacher were not included in the coding. Qualitative analysis was conducted through coding and thematic analysis (Cohen et al., 2018). With the help of Nvivo, open codes were created, confronting written protocols and students' interventions during the discussion, which was often very useful to better interpret written protocols. Codes were refined through the analysis and unified in five themes, which were then checked with the original data. I finally tried to interpret the categories in the light of Goldin's categories. For example, we can consider the following discussion in which students confront their solutions:

> Ettore: Here, there is written on text, how split the stake [...]. Yes, but [following your method] it takes an eternity to do something like that. [...] Our [solution] is faster.

Or again

> Giorgia: [...] In my opinion, your reasoning is correct, but it's impractical, because allocating 14.4 euros to one and 9.6 to the other ...
> Niccolò: Yes, it is impractical, but it depends on who wants to split the money. [...] If they are two people who know nothing about mathematics, they wouldn't doing this [applying this solution].
> [...]
> Ettore: This [solution] holds if one of the two knows mathematics.

In deciding whether the solutions of the problem are acceptable, students consider whether the solutions are easy to carry out, or practical for the player. These reasons – which are psychological facts for the students, since they are used to effectively evaluate others' solutions – have been interpreted as being connected to the *beliefs* that the correct solution should be fast or practical. Beliefs concerned with efficiency of the solving method, or the effective possibility of implementation by the players, have been unified in the theme "Usability of the solution". These beliefs emerge in a discussion on the aspects of the problematic situation that should be involved in the resolution – for example, whether it is important that the solutions divide the stake with integer number, disregarding impractical cents. These beliefs have been hence linked with the first Goldin categories, which deal with the resemblance between mathematics model and the physical word. On the other hand, they implicitly share beliefs about the knowledge of not-mathematicians regarding mathematics (sixth category).

3. Result of analysis

Before presenting the themes of analysis, I briefly comment on the solution strategies presented by the students, without going into detail about how they were presented or justified. All groups proposed to divide the stakes directly proportional to the score difference between the two players. However, the groups sustained the solutions with different arguments. The arguments differed greatly in the presence or absence of representations, such as tables; in the mathematical concepts they referred to, such as the use of proportions; and in the presence or absence of an algebraic generalization. In addition, Group 3 presented an additional solution, according to which the stakes are divided directly proportional to the score of the players when the game is interrupted. Group 3 is the only group which presented two different solutions. Students felt their solutions were different. According to classical probability, none of the solutions would be considered correct.

From the analysis the following five themes were defined.

1. *Originality*: finding more methods, or a method very different from others is *believed* to be a strength. For instance, some students evaluate the presentation of two distinct solutions by Group 3 as "accurate" and "mathematically sophisticated." This has been interpretated as being connected with beliefs about the value of finding multiple solutions.
2. *Clarity of exposition*: in their evaluations, students consider if steps of the method were clearly stated (according to them), if in the presentation some little mistakes are or not committed, and if the language was appropriate. In other words, the category entails *beliefs* concerning with the importance of accepting only clearly and correctly stated solutions.
3. *Usability of the solution*: it involves students' *beliefs* about who should use the solution, and about how fast, practical or easy the solution must be. For example, the fact that the proposed solution involved a division of the stakes with not-integer numbers was criticized because the solution was felt impractical.
4. *Generalization*: it concerns *beliefs* about the fact that a solution should be valid for all the possible outputs of the game and the generalization should be expressed in algebraic language.
5. *Validity in case of tie*: students *believed* that in case of a tie players should receive the same amount, that is, 12 euros to A and 12 euros to B. This belief was used to check the different solutions.

All the themes were considered in relation to both strengths and weaknesses, except for Originality, which was only considered in relation to strengths. The themes coexist in the same evaluation of a solution and its argument. Beliefs about the Usability of a solution, and beliefs about the Generalization of a solution, were often contrasted and created tension in the discussion. Since they were strictly connected with the evaluations of solutions and their arguments, I consider all the beliefs themes linked about effective mathematical reasoning methods and strategies or heuristics, (fourth category in Goldin's classifications). In addition, the theme of Originality can be related to beliefs about mathematical facts (second category), since students consider proportion to be a "peculiar" method, and about mathematical ability (ninth category), since finding two methods is considered by some students to be related with a higher mathematical ability. The theme Clarity of exposition can be connected to how present a solution in mathematical classroom, and therefore connected to the teaching and learning (tenth category). Moreover, the theme Usability of the solution might be linked to beliefs about which aspects of the problematic situation should be relevant in a solution (first category), and to beliefs about how mathematics is viewed by non-mathematicians (sixth category). Finally, Generalization and Validity in case of tie relate to methods to establish validity in mathematics (third category), since they were used to check the validity in all the cases or in a specific case.

4. Discussion and conclusion

Several beliefs emerge in the argumentative process led to find a common solution of the problem of the point. Students *relay on their beliefs* to assess others' solutions and the arguments presented to support it. Some solutions do not *connect with peers' beliefs* and therefore they are criticized. In other words, *beliefs* are used as *reason to explain*. However, not all beliefs that have emerged are generally considered to be *justificatory reasons* to accept or refuse a solution and the connected argument in a mathematical classroom. The individual beliefs shared by the students sometimes are in contrast with what is overall accepted as a justificatory reason in mathematics. For example, students' beliefs about the fact that the solution should be easily carried out by anyone (Usability) are in probability context generally not relevant and are usually not accepted as a justificatory reason to reject a solution. Moreover, beliefs about Originality and Clarity of Expositions are generally not alone considered to accept or refuse a solution in a mathematical classroom. However, these beliefs are crucial in shaping the idea of how a correct solution and arguments should appear in the classroom, and their discussion and negotiation, in my opinion, enrich discussion. On the contrary, the fact that a solution should be valid for all the cases considered can be linked to techniques that are widespread in the practice of mathematics to validate a statement, and hence could be considered as a justificatory reason, in this context. Moreover, the theme Validity in case of tie is consistent with the classic probability theory, since it relates to the fact that the two players in case of tie have the same probability of winning. Therefore, also if in this moment students do not know what *probability* means, the fact that, in case of tie, a solution does not give the same amount to the players can be considered as a good justificatory reason to exclude that solution.

To sum up, students beliefs play a role in assessing peers' solutions and the connected arguments. Beliefs emerged and influenced the argumentative process, since they were used to sustain or criticise the solutions. The confrontation with Goldin's categories allows us to reflect about the very different content of beliefs which emerge in this kind of activity. In fact, seven out of eleven categories were connected to the five themes, underlining the variety of students beliefs that played a role in this activity. The analysis then shows the richness of the beliefs which played a role in a classroom argumentative process, and hopefully open the to a deeper investigation of the affective aspects in the argumentation in mathematical classrooms.

Students chose the best solution as the one that had the most strengths and the fewest weaknesses. In simpler terms, the beliefs underlying their evaluations seemed to be on an equal footing. Moreover, every solution was considered to have some strengths;

therefore, none of the solutions was completely rejected. This can be related to the fact that all the proposed solutions were considered Valid in case of tie and Generalizable, and that the beliefs related to Originality and Clarity of Expositions were not felt sufficient to reject any solution. Moreover, the beliefs related to Usability were not recognized as a *justificatory reason* by the teacher. In other words, students did not reject the different solutions maybe because they did not find beliefs socially recognized as valid justificatory reasons. It is likely, however, that motivations related to social aspects and to the idea that is the teacher who should judge the correctness or incorrectness of a solution were influential.

As already mentioned, in the discussion, students' beliefs and generally accepted justificatory reasons to accept or refuse a solution in a mathematical classroom, and the connected argument, were not coincident. The teacher's role has been fundamental in smoothing out these differences. Only with her help, did students shifted their focus from those facts not considered good reasons in mathematics, such as whether it is easy for the players to divide the stakes into non-integer numbers, to reasons accepted in mathematics. Teacher's interventions were crucial in helping students find and recognize beliefs that could be considered justificatory reasons (in this case) and to reject all the solutions proposed. In other words, it seems that teacher intervention dealt with students' beliefs, and influenced them. It is recognized in literature (Yackel & Rasmussen, 2002) that social processes of teaching and learning influences the beliefs that students hold. However important, further investigation are needed to clarify the role of the teacher in such activities. This is a goal of future studies.

The study has some limitations. Only a classroom activity has been carried out, while other situations could be involved to broaden the exploration of different themes of beliefs. Moreover, reasons explicated by students have been considered as indicators of beliefs, but this aspect is sensitive, and needs further investigation. Then, in addition to beliefs, other affective aspects could be influential in argumentative processes, and they need to be explored through ad-hoc studies. Finally, scholars generally agree that beliefs are interrelated with knowledge, and some consider it as part of *subjective knowledge* (Pehkonen & Pietilä, 2003). This interconnection has not been investigated in the study. I chose to primarily link the psychological facts to students' beliefs; however, the influence of *objective knowledge* could be considered as well.

Despite several limitations, I hope this explorative provide insights into how students' beliefs play an effective role in accepting or refusing other solutions and the connected arguments, and in influencing argumentative processes even in mathematical classroom. Therefore, in an activity aimed to reach consensus about a classroom solution and to develop arguments addressed to peers, we should reflect on how helping teachers and students to observe and deal with their beliefs.

Research ethics

Artificial intelligence

Artificial intelligence has not been used in this research.

Informed consent statement

Informed consent was obtained from all research participants.

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Conflicts of Interest

The authors declare no conflicts of interest.

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